THESE de DOCTORAT DE L’UNIVERSITÉ DE LYON
OPÉRÉE AU SEIN DE L’ÉCOLE CENTRALE DE LYON

ÉCOLE DOCTORALE MEGA
Mécanique, Energie, Génie civil et Acoustique

Spécialité: Mécanique appliquée

Soutenue le 14/11/2017 par

Kaijun YI

Controlling Guided Elastic Waves Using Adaptive Gradient-Index Structures

Devant le jury composé de:

Jean-François Deü Professeur, LMSSC, CNAM Rapporteur
Massimo Ruzzene Professeur, Georgia Tech, USA Rapporteur
Olivier Poncelet Chargé de Recherche, CNRS, Examinateur
Université de Bordeaux
Emeline Sadoulet-Reboul Maître de Conférence, Examinateur
Université de Franche-Comté
Mohamed Ichchou Professeur, LTDS, ECL Examinateur
Manuel Collet Directeur de Recherche, CNRS, ECL Directeur
To my father Dehua Yi and my sister Hui Yi.
Acknowledgements

I still remember the first day I arrived in France, Dr. Changwei Zhou, Dr. Yu Fan, A-Huang (Dr. Xingrong Huang) and Dr. Xiangyun Sun welcomed me in front the main gate of ECL. Now, three years have passed. During these years, in the laboratory LTDS, I have learned a lot and received help from many kind people. Taking this opportunity, I would like to express my deepest thanks to all the people who helped and supported me during my stay in France.

First of all, I would like to express my deepest gratitude to Dr. Manuel Collet, the advisor of my thesis, for his patient and valuable guidance, sharing of many magnificent ideas. Very special gratitude to Prof. Mohamed Ichchou, for the discussions on the first part of my work. I would like to thank Dr. Sami Karkar, for the many constructive suggestions he gave to me. I would also like to express my thanks to Dr. Simon Chesné and Dr. Méloïde Monteil from INSA Lyon, for their help on the energy harvesting part of my work. Special mention to Prof. Lin Li, the supervisor of my master’s thesis, for her kind help during the application stage of this PhD. project. Besides, I sincerely thank the China Scholarship Council for financially supporting me to pursue my doctorate in France.

I am also grateful to all my colleagues. They supported me and always willing to help me. Particularly, I would like to thank Ms. Helene Schoch and Ms. Isabelle Tixier, for their help on the administrative affairs. I am also appreciate the technical support I have received from Dr. Olivier Dessombz.

And finally, last but by no means least, I would like to thank my father, my sister and other family members. Their support is very important to me. I would also like to thank all my friends in ECL, I feel really lucky to have met so many wonderful person, you make my life colorful and my experiences unforgettable.

Thank you very much, everyone!

Kaijun Yi
Ecull, November 14, 2017
Abstract

GRAdient INdex (GRIN) media are those whose properties smoothly vary in space or/and time. They have shown promising effects in many engineering applications, such as Structural Health Monitoring (SHM), vibration and noise control, energy harvesting, etc. On the other hand, piezoelectric materials provide the possibility to build unit cells, whose mechanical properties can be controlled on-line. Motivated by these two facts, adaptive GRIN structures, which can be realized using shunted piezoelectric materials, are explored in this dissertation to control guided elastic waves. Two types of adaptive GRIN structures are studied in this work.

The first type is a piezo-lens. It is composed of shunted piezoelectric patches bonded on the surfaces of plates. To control the mechanical properties of the piezoelectric composite, the piezoelectric patches are shunted with Negative Capacitance (NC). By tuning the shunting NC values, refractive indexes inside the piezo-lens are designed to satisfy a hyperbolic secant function in space. Numerical results show that the piezo-lens can focus waves by smoothly bending them toward the designated focal point. The piezo-lens is effective in a large frequency band and is efficient in many different working conditions. Also the same piezo-lens can focus waves at different locations by tuning the shunting NC values. The focusing effect and tunable feature of piezo-lens make it useful in many applications like energy harvesting and SHM. The former application is fully discussed in this thesis. The focusing effect at the focal point results in a known point with high energy density, therefore harvesting at the focal point can yield more energy. Besides, the tunable ability makes the harvesting system adaptive to environment changes.

The second type is the time-space modulated structure. Its properties are modulated periodically both in time and space. Particularly, the modulation works like a traveling wave in the structure. Due to the time-varying feature, time-space modulated structures break the reciprocity theorem, i.e., the wave propagation in them is nonreciprocal. Many unusual phenomena are observed during the interaction between waves and time-space modulated structures: frequency splitting, frequency conversion and one-way wave transmission. Two types of frequency conversion are demonstrated and explained. The first type is caused by energy transmission between different orders Bloch modes. The second type is due to the Bragg scattering effect inside the modulated structures. The one-way wave transmission could be exploited to realize one-way energy insulation in equivalent infinite or semi-infinite systems. However, the one-way energy insulation fails in finite systems due to the frequency conversion phenomenon.

Keywords: wave control, gradient index media, piezoelectric materials, negative capacitance, wave focusing, energy harvesting, nonreciprocity, frequency conversion, one-way energy insulation
Résumé

Les matériaux à gradient d’indice de réfraction (GRIN) présentent des propriétés mécaniques variant en temps ou/et en espace. Ils ont été testés pour des applications prometteuses dans de nombreuses applications d’ingénierie, comme pour le contrôle santé structurale ou la surveillance de structure (SHM), le contrôle des vibrations et bruit, la récupération d’énergie, etc. D’un autre côté, les matériaux piézoélectriques offrent la possibilité de réaliser des cellules composites dont les propriétés mécaniques peuvent être contrôlées en ligne. Motivé par ces deux approches, cette thèse étudie la mise en œuvre de structures GRIN adaptatifs pour le contrôle des ondes élastiques. Deux types de structures GRIN adaptatifs sont étudiés dans ce travail.

Le premier exemple concerne la mise en œuvre d’une lentille piézoélectrique dans une plaque. Il est composé de patches piézoélectriques shuntés, collés périodiquement en surface du guide d’ondes. Les circuits de shunt utilisés permettent d’émuler une capacité négative (NC). En accordant les valeurs de NC on peut ajuster les indices de réfraction du milieu à l’intérieur de la lentille piézoélectrique et pour satisfaire une fonction hyperbolique. Les résultats numériques montrent que les lentilles piézoélectriques peuvent alors focaliser les ondes de flexion de la plaque sur les points focaux. La lentille piézoélectrique est efficace dans une grande bande de fréquences et efficace dans une grande plage de fonctionnement. Ainsi elle peut focaliser des ondes sur différents points par simple ajustement des valeurs de NC réalisées par le circuit. Cette focalisation adaptatif la rend très intéressante pour de nombreuses applications comme la récupération d’énergie ou le SHM. La mise en œuvre de ces techniques pour la récupération d’énergie est discutée dans cette thèse.

Le second exemple concerne l’étude d’une structure dont les propriétés mécaniques sont contrôlées en temps et en espace. En particulier, une modulation périodique permet de créer une onde artificielle se propageant dans la structure. L’interaction avec des ondes mécaniques entraîne une rupture de réciprocité visible dans une diagramme de bande non symétrique. De nombreux phénomènes inhabituels sont observés dans ce type de structures variables : fractionnement des fréquences, conversion d’ondes et transmission unidirectionnelles. Deux types de conversion fréquentielle sont démontrés et expliqués. Le premier est induits par la transmission d’énergie entre les différents modes Bloch et le second type est dû à la diffusion de Bragg dans les structures modulées. La transmission unidirectionnelle des ondes pourrait être exploitée pour réaliser des dioïdes dans des systèmes infinis ou semi-infinis. Cependant, la transmission unidirectionnel n’existe pas dans les systèmes fini en raison des phénomènes de conversion de fréquence.

Mots-clés: contrôle des ondes, matériaux à gradient, matériaux piézoélectriques, capacité négative, focalisation des ondes, récupération d’énergie, non-réciprocity, conversion de fréquence, diode
Contents

Introduction xi

1 Controlling waves with gradient-index media 1
   1.1 Gradient-index lens ............................................. 1
   1.2 Acoustic black hole ........................................... 5
      1.2.1 Zero reflection effect .................................. 5
      1.2.2 One-dimensional acoustic black hole ................... 6
      1.2.3 Two-dimensional acoustic black hole .................... 7
      1.2.4 Other acoustic black hole configurations and applications 9
   1.3 Cloak ............................................................. 12
      1.3.1 Transformation method .................................. 13
      1.3.2 Cloak for acoustic waves ................................. 14
      1.3.3 Cloak for elastic waves .................................. 17
      1.3.4 Cloak for flexural waves in plates ..................... 20
      1.3.5 Remarks .................................................... 23
   1.4 Time-space modulated media ................................. 24
      1.4.1 Nonreciprocal wave propagation .......................... 25
      1.4.2 Other special phenomena induced by time-space modulated media ....... 26
   1.5 Conclusions ..................................................... 27

2 Tunable ability of shunted piezoelectrics 29
   2.1 Overview of piezoelectric materials ........................... 29
   2.2 Introduction to negative capacitance .......................... 31
   2.3 Effective material parameters of piezoelectrics shunted with negative capacitance ................................................. 33
   2.4 Dynamical properties of unit cells containing piezoelectrics shunted with negative capacitance ............................... 35
      2.4.1 Effective medium model .................................. 35
      2.4.2 Discussion .................................................. 38
   2.5 Recent advances in wave control using piezoelectric materials ...... 39
      2.5.1 Adaptive metamaterials ................................... 39
      2.5.2 Hybrid dispersive media .................................. 40
      2.5.3 Reconfigurable structures .................................. 40
      2.5.4 Gradient index interfaces .................................. 41
   2.6 Conclusions ..................................................... 42
3 Flexural waves focusing through shunted piezoelectric patches 45

3.1 Piezo-lens for wave focusing .............................................. 46

3.2 Numerical model and energy analysis .................................... 48

3.2.1 Finite element model of piezo-mechanical systems ............... 48

3.2.2 Energy analysis for harmonically excited thin plate .............. 49

3.3 Numerical results ............................................................ 49

3.3.1 Focusing effect .............................................................. 49

3.3.2 Adaptive ability ............................................................ 52

3.3.3 Performances of piezo-lens at different frequencies .......... 53

3.3.4 Performances of piezo-lens for flexural waves excited by different types of sources ............................................. 58

3.3.5 Double piezo-lenses configuration .................................... 64

3.4 Conclusion and discussion .................................................. 67

4 Enhancement of elastic wave energy harvesting using adaptive piezo-lens 69

4.1 Introduction ................................................................. 69

Part A. Concept of exploiting piezo-lens in wave energy harvesting .... 71

4.2 Full finite element model .................................................... 73

4.3 Wave energy harvesting performances of the system incorporating the piezo-lens ..................................................... 74

4.3.1 Enhancement of the harvested power ................................ 74

4.3.2 Adaptive feature ............................................................ 77

Part B. Harvesting energy from transient elastic waves using the piezo-lens and SSHI-based techniques ............................................ 79

4.4 Harvesters ................................................................. 79

4.5 Reduced finite element model .............................................. 82

4.6 Numerical results ............................................................ 85

4.6.1 Comparison between the standard DC and SSHI-based devices 85

4.6.2 Effects of piezo-lens on transient waves and harvesting ....... 87

4.6.3 Energy balance ............................................................... 90

4.6.4 Practical application considerations ................................... 92

4.7 Conclusions of Part A and B ................................................ 94

5 Wave propagation in time-space modulated structures 97

5.1 Time-space modulated beam ................................................. 97

5.2 Dispersion relations and Bloch modes in time-space modulated beams 98

5.2.1 Analysis of dispersion .................................................... 98

5.2.2 Band diagrams .............................................................. 99

5.2.3 Bloch modes as group of harmonics ................................ 103

5.3 Frequency conversion ...................................................... 105

5.3.1 Theory ................................................................. 106

5.3.2 Results ................................................................. 109

5.3.3 Numerical simulations ................................................... 113
5.4 Conclusions ................................................................. 115

6 Reflection and transmission of elastic waves incident on time-space modulated structures ........................................... 119
   6.1 Bloch modes in modulated beams .................................. 120
   6.2 Reflection and transmission of elastic waves incident on modulated beams ................................................. 121
      6.2.1 Introduction to the scattering matrix method ............... 121
      6.2.2 Extending the scattering matrix method to modulated beams .......................................................... 121
      6.2.3 Energy flux ....................................................... 124
   6.3 Results ...................................................................... 125
      6.3.1 Validation of the extended scattering matrix method ....... 125
      6.3.2 Frequency reflection and transmission properties ........... 127
      6.3.3 Insights into the unusual phenomena induced by modulated beams ...................................................... 129
      6.3.4 Influences of the modulation velocity ......................... 132
      6.3.5 Energy balance analysis .......................................... 134
      6.3.6 The feasibility of one-way energy insulation in finite systems ............................................................ 139
   6.4 Conclusions ............................................................ 142

Conclusions and perspectives ............................................ 145

A Material parameters of the aluminum and PZ26 .................... 147

B Fully coupled finite element models of piezoelectric systems and Bloch boundary conditions ........................................ 149

C Supplementary results for Chapter 3 .................................. 153

D Validation of the corrected reduced models of piezoelectric systems ................................................................. 157

E Details of matrices in Chapter 6 ........................................... 161

F Proof of the periodicity of Equations (6.13) and (6.14) in Chapter 6 ................................................................. 163

List of abbreviation .................................................................. 165

List of Figures ..................................................................... 167

List of Tables ...................................................................... 177

List of publications .............................................................. 179

Bibliography ........................................................................ 181
Introduction

Motivation

Elastic wave propagation in structures is of great interest to understand and control vibration and noise, especially in large size structures composed of a number of components. The effect of a sharply applied, localized disturbance in some part of assembled (built-up) structures, like vehicles, civil architectures, aircrafts and spaceships, etc., transmits in the form of waves to other components of the structures, consequently, spreading energy from parts to parts. Inevitably, waves encounter and interact with boundaries, results in reflection and transmission at the boundaries. Specially, the continual propagation and reflection in bounded structures brings about the state of static equilibrium, even the resonance. Besides, interaction between structures in which waves are propagating and fluids on one hand causes radiation of noise into the fluid, on the other hand results in transmission of noise from the interior to exterior or vice versa. Nevertheless, waves are not always undesirable in structures. Artificially excited waves are often used to detect flaws in structures or to monitor their health.

The above mentioned strong link between the wave propagation and many engineering issues, as well as the practical applications by exploiting waves stimulate interests in controlling waves in structures. Among the efforts, GRadien-t-INdex (GRIN) media made significant contributions. GRIN is often used in optics to describe inhomogeneous media in which the refractive index varies from point to point [1]. This terminology is followed and extended to media with time-space varying properties in this thesis. In GRIN media, waves can follow curved trajectories, they can also speed up or slow down. Therefore, by smoothly changing the properties in space or modulating them both in space and time in an appropriate way, we can design GRIN media that can lead to many effects, such as wave focusing, trapping, cloaking and nonreciprocal propagation. These fascinating phenomena broaden the way we control waves in acoustics and mechanics. They already found applications or may find in structural health monitoring, energy harvesting, acoustic and vibration control, etc.

In addition, smart materials were widely used to control waves for many years. Among the smart materials, piezoelectric materials have attracted intense attention for their virtues such as compact size and high coupling factor. As first reported by Hagood and Von Flotow [2], one knows that the effective material properties of piezoelectric patches can be changed by tuning the shunting impedance. Specially, when the shunt is a negative capacitance (NC) circuit, the effective Young's modulus of the piezoelectrics is independent of frequency to some extent, and it can be varied within a very large range. These virtues have already been exploited to build metamaterials with tunable band gaps [3] or with reconfigurable cell symmetry [4].
The many practical applications of GRIN media and the ability of real-time modifying structural properties using piezoelectric materials motivate us to explore adaptive GRIN structures to control wave propagation. These structures have the capabilities to control their interior energy distributions, vibrations, noise radiation, etc. Also they are adaptable to environment changes therefore are more reliable in reality and can meet the growing human needs of more advanced equipments.

Outline of the thesis

In this thesis, we study two types of adaptive GRIN structures. The first one is called piezo-lens used to focus flexural waves in plates. It is composed of piezoelectric patches bonded on the surfaces of plates and shunted with NC circuits. The refractive index inside the piezo-lens is designed to fulfill a hyperbolic secant function by tuning the shunting NC circuit values. Our attention is payed to the flexural waves for they are the most easily excited modes in plates and often carry most of the energy [5]. Studies of the piezo-lens are presented in Chapters 3 and 4. The second type is the time-space modulated structure which reveals many attractive properties such as nonreciprocity. This type of structure is obtained by modulating the properties of the structure both in time and space. More specifically, the modulation acts like a traveling wave in the structure. This kind of modulation could be realized by using piezoelectric materials. Studies of this part are presented in Chapters 5 and 6. Outlines of each chapter are listed below.

Chapter 1 introduces efforts made in controlling waves using GRIN media. Special attention is paid to GRIN lens, acoustic black hole, cloak and time-space modulated media. Concepts, theories, performances and applications of these GRIN devices are fully introduced and discussed.

Chapter 2 demonstrates the tunable ability of piezoelectrics shunted with NC. General properties of piezoelectric materials and concepts of NC circuits are introduced first. Then, the effective material parameters of piezoelectrics shunted with NC is obtained. Further more, an Effective Medium Model (EMM) is developed for unit cells containing piezoelectrics shunted with NC. Tunable feature of the dynamical properties of the cells is presented using the EMM, the accuracy of the EMM is discussed. Recent advances in wave control using piezoelectric materials is also covered in this chapter.

Chapter 3 designs and studies the piezo-lens for focusing flexural waves in plates. Concept and designing process of the piezo-lens are introduced. Finite Element (FE) method is used to study the performance of the piezo-lens. The focusing effect of the piezo-lens is verified. Adaptive feature of the piezo-lens is demonstrated. Effective frequency band of the piezo-lens is estimated. Feasibility of the piezo-lens in several different cases is evaluated. A double lens configuration is also proposed, effect of which is well confirmed.

Chapter 4 explores the piezo-lens in harvesting energy from traveling waves. This chapter is divided into two parts. Part A introduces the concept of using piezo-lens in
wave energy harvesting. Enhancement of the harvested power from traveling waves is verified, also the adaptive ability of the harvesting system incorporating the piezo-lens to environment changes is demonstrated. Part B further combines the piezo-lens with Synchronized Switch Harvesting on Inductor (SSHI) techniques to yield energy from transient waves. Two different SSHI-based harvesters are used. The focusing effect of the piezo-lens for transient waves and the harvesting performance of the proposed systems are studied using corrected reduced FE models.

Chapter 5 is dedicated to the wave propagation in time-space modulated structures. Properties of the dispersion relations and Bloch modes in these beams are studied by comparing with those in homogeneous or space-only periodic beams. The frequency conversion phenomenon induced by time-space modulation is theoretically and numerically demonstrated. Two types of frequency conversion are observed. They underlying mechanisms are explained in details.

Chapter 6 studies the properties of reflection and transmission when elastic waves are incident on time-space modulated structures. The scattering matrix method is extended to study the time-space modulated structures. The corresponding scattering matrix is developed. Using this matrix, the nonreciprocal transmission, frequency splitting and frequency conversion caused by the time-space modulation are studied and the mechanisms behind these unusual phenomena are interpreted. Influences of the modulation parameters are evaluated as well as the energy balance. The feasibility of using time-space modulated structures in building one-way energy insulators is discussed.

Finally, main contributions of this thesis are summarized and the future work for adaptive GRIN structures is briefly discussed.
Chapter 1

Controlling waves with gradient-index media

Contents

1.1 Gradient-index lens ........................................ 1
1.2 Acoustic black hole ........................................ 5
  1.2.1 Zero reflection effect ................................. 5
  1.2.2 One-dimensional acoustic black hole ............... 6
  1.2.3 Two-dimensional acoustic black hole ............... 7
  1.2.4 Other acoustic black hole configurations and applications .. 9
1.3 Cloak ..................................................... 12
  1.3.1 Transformation method .............................. 13
  1.3.2 Cloak for acoustic waves ......................... 14
  1.3.3 Cloak for elastic waves ........................... 17
  1.3.4 Cloak for flexural waves in plates ............... 20
  1.3.5 Remarks .............................................. 23
1.4 Time-space modulated media .............................. 24
  1.4.1 Nonreciprocal wave propagation ................... 25
  1.4.2 Other special phenomena induced by time-space modulated media ........................................ 26
1.5 Conclusions ............................................... 27

In this chapter, the state of art of the implementations using GRIN media in controlling acoustic and elastic waves is introduced. Special attention is paid to GRIN lens (Section 1.1), acoustic black hole (Section 1.2), cloak (Section 1.3) and time-space modulated media (Section 1.4).

1.1 Gradient-index lens

Wave focusing is promising to be useful in applications such as Structural Health Monitoring (SHM) [6], energy harvesting [7, 8, 9], acoustic imaging [10, 11, 12, 13, 14, 15], etc. Within the last decade, focusing acoustic and elastic waves was realized using GRIN media.
The first success of focusing acoustic wave using GRIN media was made by S. Lin et al. [16]. In their proposition, a flat lens, as shown in Figure 1.1, is obtained by varying the refractive index in the transverse direction (namely, direction $y$ in Figure 1.1) according to a hyperbolic secant function:

$$n(y) = n_0 \cdot \text{sech}(\alpha y) \quad (1.1)$$

in which, $n_0$ is the refractive index along the center axis ($x$ axis), $\alpha$ is the gradient coefficient, which determines the focal length:

$$f = \frac{\pi}{2 \alpha} \quad (1.2)$$

The mechanism behind the focusing is the bending of the wave trajectory and delay introduced by the variation of the refractive index. The locations near the center axis have larger refractive indexes. Therefore, waves incident into the medium along the $x$ axis will be bent toward the center axis due to the refraction. Also as the distance from the center axis increases, the wave speed will gain. Consequently, waves incident at different $y$ locations at the left entrance of the lens propagate over different distances but simultaneously arrive at the focal point with the same phase, resulting in a spot with high energy density.

Focusing waves can be obtained using different mechanisms. A significant advantage of GRIN lenses compared with those using negative refraction [10, 11, 12] is that their focal points are easy to be controlled. As indicated in Equation (1.2), the focal length is only determined by the gradient coefficient $\alpha$. Therefore, by choosing different $\alpha$ values, we can focus the waves at desired locations, even outside the lens.

The GRIN lens to focus waves was realized by using Phononic Crystals (PCs). PCs are composites with periodically arranged unit cells. They behave at low frequency band as homogeneous materials with effective parameters mainly depend on the unit cell [17]. Therefore, by locally changing the parameters of the unit cell,
the variation characterized by Equation (1.1) can be approximately fulfilled in a piecewise form. As shown in Figure 1.2, the GRIN lens can be obtained by varying the filling fraction (left figure) or the filling materials (right figure) of the PCs to satisfy the variation of the refractive index described by Equation (1.1).

![Figure 1.2](image)

Figure 1.2: A GRIN lens is obtained by (left) adjusting radii of cylinders or (right) changing elastic properties of cylinders along the transverse direction [16].

Following S. Lin et al.'s work [16], GRIN lenses based on PCs were designed for acoustic waves [18, 19] and guided waves in plate [7, 20, 21, 8, 9]. Besides, The GRIN lens was also realized by using metamaterials. Metamaterials exhibit unusual effective dynamic properties [22, 23, 24, 25], which make them alternative candidates to achieve GRIN lenses. For example, in the work of X. Yan et al [6], the GRIN lens is realized by using the metamaterials composed of cells illustrated in Figure 1.3, which contain lead discs bonded on aluminum plates using uniform thickness silicone rubbers.

![Figure 1.3](image)

Figure 1.3: One cell of the metamaterial for GRIN lens. The top part is a lead disc, the central part is a silicone rubber and the low part is an aluminum plate [6].

The effective mass density of the cell in Figure 1.3 at a particular frequency can be tuned by changing the thickness of the lead discs (the top dark part). According to this, a GRIN lens was realized by periodically arranging the cells on the plate surface and tuning them to fit the required refractive index variation, as shown in Figure 1.4. A major drawback of the GRIN lenses made by metamaterials is that they can only work in a very narrow frequency band.

Except the flat GRIN lenses as shown in Figures 1.1 and 1.4, there are also circular ones [26, 27] for focusing flexural waves in plates. The phase and group
velocities of flexural waves in plates are functions of the plate thickness $h$. It is straightforward to obtain the refractive index for those waves also as a function of the thickness [26]:

$$n(r) = \sqrt{\frac{h_b}{h(r)}}$$  \hspace{1cm} (1.3)

in which, $h_b$ is the background’s thickness and $h(r)$ is the thickness of the lens in a polar coordinate. Therefore, the GRIN refractive properties in these lenses were usually obtained by changing the thickness of the lens, as shown in Figure 1.5.

In addition, the GRIN lens was also realized by using metasurfaces, which can significantly reduce the lens’s volume. Figure 1.6 shows the metasurface proposed by Y. Li et al. [28]. The acoustic impedance of the metasurface was designed to satisfy a function $Z_s(y)$, which is determined according to the requirement. For example, to focus the sound at $(0.3, 0)$ emitted by a source at $(1, 0)$, the impedance...
1.2. Acoustic black hole

$Z_s(y)$ of the metasurface need to be designed as:

$$Z_s(y) = \frac{-i\rho_0 c_0}{\tan(\phi(y)/2)}, \quad \phi(y) = -k_0(\sqrt{y^2 + 0.9} - 0.3) - k_0(\sqrt{y^2 + 1} - 1)$$

(1.4)

in which, $i = \sqrt{-1}$, $\rho_0$ and $c_0$ are the density and sound velocity in air, respectively.

![Figure 1.6: The metasurface and schematics for the derivation of the acoustic focusing with focus length $f$ [28].](image)

1.2 Acoustic black hole

1.2.1 Zero reflection effect

A zero reflection effect or Acoustic Black Hole (ABH) can be achieved by smoothly diminishing the phase and group velocities of waves to zero. Considering an one-dimensional wave propagation in an ideal medium with power-law dependence of wave velocity $c$ on $x$ as [29]:

$$c(x) = ax^n$$

(1.5)

here, $a$ and $n$ are all constant numbers. The time needed for the wave to travel from an arbitrary point $x_i$ to another point $x_j$ in the medium can be derived from:

$$T = \int_{x_i}^{x_j} \frac{1}{c(x)} \, dx$$

(1.6)
and be expressed as:

$$T = \frac{1}{a} \left( \frac{1}{x_j^{n-1}} - \frac{1}{x_i^{n-1}} \right)$$  \hspace{1cm} (1.7)$$

From Equation (1.7) it is obvious that when \( n > 1 \) and \( x_j \) tends to zero, where the wave velocity is zero, the time needed for the wave to travel to the zero point is infinite. In other words, under these circumstances the wave can never reach the zero point, which means the wave will never be reflected back and it becomes trapped, implying that the aforementioned ideal medium can be consider as an acoustic black hole for the wave under consideration.

Exploiting the zero reflection effect, ABH has been mainly investigated for flexural waves (namely, the fundamental antisymmetric Lamé mode \( A_0 \)) in one-dimensional and two-dimensional structures. The smooth decrease conditions are mainly fulfilled by changing the structure thickness \( h(\tau) \) according to a power-law principle [30]:

$$h(\tau) = \varepsilon x^m$$ \hspace{1cm} (1.8)$$

in which, \( \varepsilon \) is the scalar factor, \( m \) is a constant which must satisfy \( m \geq 2 \).

### 1.2.2 One-dimensional acoustic black hole

M. Mironov [31] was the first one who realized this zero reflection effect at the edge of a plate, whose thickness near the edge smoothly decreases to zero, as shown in Figure 1.7. However, practical realization of such power-law profile is impossible since truncation always exists at the tip of the structure. M. Mironov pointed out that even very tiny truncation will cause high reflection coefficient enough to make this technique useless.

![Figure 1.7: Plate with a power-law tapered edge. Ideally, the plate should extend to \( x = 0 \) with a zero thickness at that point. However, due to the inevitable truncation in practice, the plate is truncated at \( x = x_1 \) [31].](image)

V. Krylov and F. Tilman [30] revisited the zero reflection effect more than 2 decades later after being proposed by M. Mironov. They studied the wave reflection from the elastic wedge of power-law profile shown in Figure 1.8. They confirmed the conclusion of M. Mironov that very small truncation of the edge will lead to large reflection. However, they found that, adding thin absorbing films on the wedge
surfaces can significantly reduce the reflection of waves from the truncated edges due to the enhanced energy absorption by such films.

![Graph](image)

Figure 1.8: The elastic wedge of power-law profile studied in [30].

V. Krylov and F. Tilman’s work [30] triggered the enthusiasm of many researchers to study the ABH. Within the last 2 decades, many theoretical, numerical and experimental works have been done. These researches mainly focused on the following aspects:

- Developing theoretical models for the ABH. M. Mironov [31] applied the Wentzel-Kramers-Brillouin (WKB) approximation to obtain the displacement of the plate in Figure 1.7 and then deduced the reflection coefficient from the truncated edge. V. Krylov and F. Tilman [30, 29] introduced absorption layers into the ABH, therefore they used the model from Ross-Ungar-Kerwin [32] to analyze the influence of the layer on the wave propagation. This model doesn’t take into account the evanescent waves. Georgiev et al. [33] proposed a model that takes into account the attenuating part of the wave field to compute the reflection coefficient of an ABH beam using an impedance matrix technique.

- Applying the ABH in vibration control. The ABH accumulates energy at the edge and dissipates it using the absorption layer. Therefore, the ABH is attached to beams and plates’ edges to attenuate structural vibration [34, 35, 36, 37, 38]. The ABH can significantly reduce the vibration level in a large frequency band. V. Denis et al. [37] found that the attenuation is caused by a strong increase of the modal loss factors. The absorption layer’s parameters also have important influences on the vibration control effect [38]. Usually, a thick layer will lead to better damping effect. Also better vibration reduction effect will be achieved as the layer is attached more close to the tip of the ABH.

### 1.2.3 Two-dimensional acoustic black hole

The same physic principle of flexural wave propagation in structure with power-law profile edge can be applied for two-dimensional acoustic black hole. Unlike
one-dimensional cases, in which waves slow down smoothly when propagate toward the edge, in two-dimensional cases waves not only slow down but also be deflected smoothly.

A typical two-dimensional acoustic black hole structure is shown in Figure 1.9(a), the pit drilled in the regular plate has power-law profile. In the pit area refraction index of flexural wave is:

\[
n(r) = \left[ \frac{h_0}{h(r)} \right]^{1/2}
\]

in which, \( h_0 \) is thickness of the plate outside of the pit area and \( h(r) = \varepsilon r^m \) is local thickness of the pit area. From Equation (1.9) it is obvious that refraction index in pit area varies symmetrically and smoothly, the position of the maximum value is in the central of the pit. The analysis in [39] shows that with \( m \geq 2 \), rays that are close enough to a direct ray, including a direct ray itself will deflect toward the central of the pit, approaching it almost in the normal direction, as you can see in Figure 1.9(b). Since the central area of the pit is covered by absorbing layers, most of the captured rays are absorbed leading to a reduction of the incident energy.

![Figure 1.9](image)

(a) (b)

Figure 1.9: (a) A circular ABH on a plate [29]. (b) Typical ray trajectories illustrating propagation of flexural waves over a circular ABH [39].

The 2D circular ABH was embedded into plates and beams to attenuate vibration [33, 40, 41, 42, 43]. Very good attenuation of vibration has been observed especially at high frequencies. Also 2D circular ABHs were periodically arranged to control wave propagation. L. Tang and L. Cheng [44] constructed a 1D PC by repeating circular ABH in one direction, as shown in Figure 1.10(a). They found that the ABHs act like local resonators and create band gaps, the widths of which can be broadened by increasing the power index \( m \) or reducing the truncation thickness. H. Zhu and F. Semperlotti [45] studied the wave propagation in a 2D PC composed of circular ABHs shown in Figure 1.10(b). They found that, this ABH-PC provide the same wave propagation effects typically observed in locally resonant materials, including negative refraction, bi-refraction, Dirac cones, and mode hybridization.
1.2. Acoustic black hole

1.2.4 Other acoustic black hole configurations and applications

Except varying the thickness of beams and plates, the ABH effect has also been obtained by using other configurations. A. Ouahabi et al. [46] experimentally investigated the using of ABH for sound absorption in a tube. The ABH is realized by varying the inner radius according to a power-law function, as shown in Figure 1.11. Reduction of sound reflection coefficients was observed in the experiments when the ABHs were used without inserted damping materials. However, the insertion of porous absorbing materials, fiberglass and mineral cotton, didn’t bring further noticeable reductions to the sound reflection coefficients. Their concluded that further theoretical and experimental investigations should be carried out to clarify these issues and to achieve lower values of sound reflection coefficients.

Jae Yeon Lee and Wonju Jeon [47] stated that the effect of the original ABH with straight wedge type profile will be better as the length of the ABH increases. However, too long ABH will occupy too much space hence is difficult to implement in practice. Therefore, they proposed a spiral ABH to suppress vibration of plates, as shown in Figure 1.12.

By exploiting the virtue of PCs, which can mimic homogeneous materials with different parameters at sub-wavelength range, the ABH effect is also obtained [48, 49, 50]. In these studies, the region that can trap and absorb waves is called...
Chapter 1. Controlling waves with gradient-index media

Figure 1.11: (a) Schematic view of the manufactured Linear Acoustic Black Hole (LABH) showing the wooden backing and the distribution of the ribs whose inner radii decrease to almost zero. (b) Prototype of the LABH used in experiments [46].

Figure 1.12: Plate with an ABH of an Archimedean spiral shape [47].

Acoustic (omnidirectional) absorber. The mechanism of these acoustic absorbers are exactly the same as the circular ABH introduced above. Namely, they bending and concentrating waves using GRIN media, then dissipate these waves by damping materials. The difference is that, in these acoustic absorbers, the GRIN media to slow down and to bend waves are realized by changing the cells of PCs at different locations. For example, an PC-based acoustic absorber is shown in Figure 1.13, which is proposed by A. Climente et al [49].

Other applications of ABHs except attenuating flexural waves in beams and plates also have been studied. E. Bowyer and V. Krylov [51] exploited the using of ABH to damp turbofan blades. Turbofan blades under operation suffer complicated and wide band excitations from the fluid. Broadband and robust vibration control strategies therefore are essential for them. The ABH is a promising method to be used on the fans. In the proposition of E. Bowyer and V. Krylov, the edge of the fan is tailored according to the power-law profile, as shown in Figure 1.14. The change
1.2. Acoustic black hole

Figure 1.13: Photograph of the structure studied in [49]. The outer shell is made of cylinders whose diameters increase with decreasing distance to the center. The inner core is made of identical cylinders in a hexagonal lattice with about 84% of filling fraction. The inset shows the ray trajectories of the sound traveling within the outer shell.

of the edge will damage the aerodynamic performance of the fan. Therefore, they proposed to cover the edge with absorption layers, which are also necessary for the ABH, to recover the shape of the fan.

Figure 1.14: Fan blade profile with tapering according to the power-law geometry [51].

Shunted piezoelectric materials for passive damping have been studied for decades after Hagood and Von Flotow [2]. The shunted piezoelectric patches are combined with the ABH to replace the absorption layers by L. Zhao [52]. The piezoelectric patches are attached under the circular ABHs, as shown in Figure 1.15. Resistive shunts were used in his study and broad band vibration reduction was confirmed. Since the piezoelectric patches can be used to damp energy concentrated by the ABH, it is straightforward to predict that harvesting inside the ABH will be more efficient. This idea was proposed and verified by L. Zhao et al [53, 54].
1.3 Cloak

The successful achievement of cloaking devices for electromagnetic [55] and optic waves [56] based on geometry transformation as well as the subsequent experimental demonstration [56] stimulated interests in cloaking of wave motions. In the last decade, transformation based cloak theories for acoustic wave, elastic wave and flexural wave in thin plate have been established. Details of these theories are introduced and discussed hereinafter.

The fundamental property that enable objects to be cloaked from electromagnetic wave or optic wave is the invariance of the Maxwell’s equations under geometry transformation. This invariance was also fund in transformation of acoustic equations [57, 58, 59]. The acoustic cloak theory for classical fluid was proposed firstly by adopting a “push-out” transformation which results in cloaks with isotropic bulk modulus and torsorial density [60]. In this transformation, the mapping is one-to-one except the point which is expanded to form the inner boundary of the cloak. This singular property leads to that the material properties for the ideal cloak are singular at the inner boundary (e.g., infinite density), which makes the cloak impractical. Two strategies were proposed in the references to obtain more practical cloaks. The first one slightly modified the governing equation for the cloak to loose the requirement for the material properties [61]. The second one mapped the inner boundary of the cloak from a small area instead of from a single point [62]. These two strategies create approximate cloaks (or reduced cloaks) who have reflections on the inner boundaries. A more general acoustic cloak theory was developed by Norris to design acoustic cloak [59]. Materials for these acoustic cloaks are expanded from classic fluid medium to pentamode materials and are non-unique for a given transformation, infinite mass requirement can be avoid in these cloaks.

However, the elastic wave equation is not invariant under a general transformation [63, 64]. G. W. Milton et al. [63] found that the transformed elastic equation is special case of Willis equations, in which density is anisotropic, the stress not only couples with the strain but also with the velocity of the particle, and the moment gets coupled with the velocity as well as the strain. In contrary to G. W. Milton et al.’s result, M. Brun et al. [65] fund that the transformed elastic equation can possesses isotropic density but non-symmetric constitutive relations where only the major symmetry of the elasticity matrix is preserved. More general elastic cloak theory was developed by Norris et al. [64]. In their theory, geometry transformation as well as a “gauge” relating the displacement fields in the initial domain and
the transformed domain are free to designate. For a given geometry transformation, different materials for cloak can be obtained by designating different gauges. It was interpreted in [64] that G. W. Milton et al.’s result [63] is corresponding to the case that the gauge is equal to the geometry transformation matrix and M. Brun et al.’s result [65] is associated with unity gauge.

Equation for flexural wave in thin plate is also not invariant under transformation [66, 67, 68, 69, 70]. Farhat et al. [66, 67] found that the transformed plate equation can be identified as the equation of an anisotropic plate if von Kármán’s strain is assumed. However, according to von Kármán’s theory [71], membrane forces in the middle plane are associated with the out-of-plane displacement, hence, the equivalence between the transformed plate equation and the plate equation under von Kármán’s theory can only be guaranteed at particular circumstances. A more mathematically rigorous cloaking theory for flexural wave was proposed by M. Brun and D. J. Colquitt et al [69, 70]. They found that approximate cloaks for flexural waves can be obtained by only designing the material of orthotropic plate, ideal cloaks not only require GRIN materials but also need membrane forces and in-plane body forces on the middle plane of the orthotropic plate, these forces only depend on the locations.

In the rest of this section, works in [72, 55, 57, 64, 69, 70] are succinctly introduced to systematically present the general transformation method for controlling waves and details of the cloaking theories for acoustic, elastic and flexural waves in thin plates.

### 1.3.1 Transformation method

Two related domains will be considered, the initial domain (or virtual domain) \( \Gamma \) and the transformed domain (or physic domain) \( \gamma \). In what follows, symbols \( \nabla_X \), \( \nabla \) and \( (\nabla_X \cdot) \), \( (\nabla \cdot) \) indicate gradient operators and divergence operators in \( \Gamma \) and \( \gamma \), respectively; and \( \nabla_X^2 \), \( \nabla^2 \) represent Laplacian operators in \( \Gamma \) and \( \gamma \), respectively. Upper and lower case subscripts \( (I, J, \ldots, i, j, \ldots) \) are used to distinguish between these two domains. The summation convention on repeated subscripts is assumed and a comma in subscript denotes differential with respect to the coordinate variables.

Consider a physic process in the initial domain \( \Gamma \), the field variable \( U \) and material \( M_a \) are related together with a point's coordinate variable \( X \) and time variable \( t \) by a differential equation \( F \) as:

\[
F(U, M_a, X, t) = 0, \quad X \in \Gamma
\]  

The differential equation \( F \) will be specified for acoustic wave, elastic wave and flexural wave in thin plate in the subsequent sections. The art of transformation method is to perform a mapping to map the domain \( \Gamma \) to \( \gamma \) by [72]:

\[
x = T_X(X), \quad u = T_U(U), \quad m_a = T_{M_a}(M_a)
\]
These three transformation relations can be termed geometry transformation, variable transformation and material transformation, respectively.

After performing the transformations in Equation (1.11), the physical process governed by Equation (1.10) in the initial domain $\Gamma$ will be transformed to the transformed domain $\gamma$ governed by a differential equation $f$:

$$f(u, m_a, x, t) = 0, \ x \in \gamma$$  \hspace{1cm} (1.12)

By delicately designing the transformations in Equation (1.11), desired wave propagation effects can be obtained. Note that these transformations are not independent, they are related by two constraints. The first one is a geometrical constraint which relates the differential operators in the two domains:

$$\nabla_X = F^T \nabla, \ (\nabla_X \cdot) = F^T (\nabla \cdot), \ \nabla^2_X = J \nabla \cdot (J^{-1}FF^T \nabla)$$  \hspace{1cm} (1.13)

here, $T$ means matrix transpose, $F$ is the Jacobian matrix corresponding to the geometry transformation:

$$F = \nabla X x, \ F_{IJ} = \partial x_I / \partial X_J$$  \hspace{1cm} (1.14)

Its corresponding Jacobian determinant is:

$$J = \det(F)$$  \hspace{1cm} (1.15)

The other constraint is a physical constrain which guarantees the conservation of any type of energy density after the transformation:

$$E(U, M_a) = J \cdot e(u, m_a)$$  \hspace{1cm} (1.16)

The cloaking effects are examples of implementation of the transformation method introduced above. In cloaks respectively for acoustic wave, elastic wave and flexural wave in thin plate, the geometry transformation and the variable transformation are predesignated. The geometry transformation creates the cloak with desired shape, and it along with the variable transformation determine the governing equations for the cloak. Mimicking these governing equations by using GRIN materials or/and applying special conditions in the cloak region, cloaking effect can be obtained.

### 1.3.2 Cloak for acoustic waves

The acoustic equation for the scalar pressure $p(x)$ in classical homogeneous and inviscid fluid medium with bulk modulus $K_0$ and density $\rho_0$ is:

$$K_0 \nabla^2_X p + \rho_0 \omega^2 p = 0$$  \hspace{1cm} (1.17)

Adopting the geometry transformation and assuming $p(x) = p(X)$, Equation (1.17) is transformed to:

$$K_0 J \nabla \cdot (J^{-1}FF^T \nabla p) + \rho_0 \omega^2 p = 0$$  \hspace{1cm} (1.18)
Therefore, Equation (1.18) is the governing equation for the desired acoustic cloak. It has exactly the same form as the acoustic equation for fluid medium with isotropic inhomogeneous bulk modulus $K$ and anisotropic density $\rho$:

$$K \nabla \cdot (\rho^{-1} \nabla p) + \omega^2 p = 0$$  \hspace{1cm} (1.19)

Hence, the ideal acoustic cloaks can be obtained by designing the material parameters for the fluid medium in the cloak as:

$$K = K_0 J, \quad \rho = \rho_0 J (\mathbf{F}^T \mathbf{F})^{-1}$$  \hspace{1cm} (1.20)

As examples, an ideal cloak and an approximate cloak in 2D field will be designed and their effects will be verified. These 2D acoustic cloaks are degenerated from corresponding 3D ones. Consider the radial “push-out” transformation introduced in [55, 57] to create a cylindrical cloak of inner radius $R_1$ and outer radius $R_2$. Within $0 \leq R \leq R_2$ the transformation is given by:

$$\begin{align*}
    r &= R_1 + \frac{R_2 - R_1}{R_2} R \\
    \theta &= \Theta \\
    z &= Z
\end{align*}$$  \hspace{1cm} (1.21)

Its corresponding Jacobian matrix in cylindrical coordinates and Jacobian determinant are:

$$\mathbf{F} = \begin{bmatrix}
    \frac{R_2 - R_1}{R_2} & 0 & 0 \\
    0 & \frac{R_2 - R_1}{R_2} & \frac{r - R_1}{R_2} \\
    0 & 0 & 1
\end{bmatrix}, \quad J = \left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{r}{r - R_1}$$  \hspace{1cm} (1.22)

Transformation denoted by Equation (1.21) or Equation (1.22) will create a 3D cylindrical cloak, the parameters for this 3D cloak can be determined by substituting Equation (1.22) into Equation (1.20). As there is no distortion in $Z$ axis direction in the transformation denoted by Equation (1.22), a 2D ideal acoustic cloak can be directly obtained by degrading the 3D one [57]:

$$K = K_0 \left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{r}{r - R_1}, \quad \rho = \rho_0 \begin{bmatrix}
    \frac{r}{r - R_1} & 0 \\
    0 & \frac{r - R_1}{r}
\end{bmatrix}$$  \hspace{1cm} (1.23)

It can be observed from Equation (1.23) that the bulk modulus is isotropic but the density is anisotropic (denoted by an order-2 tensor), therefore the local wave velocity in the cloak can be defined as an order-2 tensor $c = \sqrt{K/\rho}$, its component form in cylindrical coordinates is:

$$\mathbf{c} = c_0 \begin{bmatrix}
    \frac{R_2 - R_1}{R_2} & 0 \\
    0 & \frac{R_2 - R_1}{R_2} & \frac{r - R_1}{r}
\end{bmatrix}$$  \hspace{1cm} (1.24)

here, $c_0 = \sqrt{K_0/\rho_0}$ is the wave velocity in the initial medium. From Equation (1.24) it is observed that the velocity in radial direction is independent with the location.
but the tangential one smoothly increases to infinite as the location approaches the inner boundary of the cloak. These characteristics of the local wave velocity in the cloak lead the waves bypass the obstacle according to the Snell’s law.

It can also be observed from Equation (1.23) that all these quantities are location dependent, the bulk modulus and the component of the density in radial direction will be infinite as the location approaches the interior boundary of the cloak. The infinite density at the inner boundary leads to infinite mass for the cloak, makes the ideal acoustic cloak impractical.

A more practical 2D acoustic cloak can be obtained by multiplying all the parameters in Equation (1.23) by $\frac{R_2(r - R_1)}{(R_2 - R_1)r}$, leading to the following parameters:

$$K' = K_0 \frac{R_2 - R_1}{R_2}, \quad \rho' = \rho_0 \left[ \frac{R_2}{R_2 - R_1} 0 \right. \left. 0 \frac{R_2}{(r - R_1)^2} \right]$$ (1.25)

Cloak indicated by Equation (1.25) can be interpreted as a degraded cloak from a 3D ideal cloak generated by the geometry transformation below in cylindrical coordinates:

$$F = \begin{bmatrix} \frac{R_2 - R_1}{R_2} & 0 & 0 \\ 0 & \frac{R_2 - R_1}{R_2} & \frac{R_2}{r - R_1} \\ 0 & 0 & \frac{R_2}{r - R_1} \end{bmatrix}, \quad J = \frac{R_2 - R_1}{R_2}$$ (1.26)

The bulk modulus as well as the component of the density in radial direction illustrated in Equation (1.25) are constants and the other component of the density is varying within a limited range. Considering the wave velocity tensor $c'$ in the redesigned cloak, obviously it has the same components as in Equation (1.24), hence the redesigned cloak also has the ability to steer the waves bypass the obstacle. The challenging for the redesigned cloak is that it can’t make all the waves bypass the obstacle, i.e., tiny part of the energy will be scattered by the obstacle. The reason can be explained as that the degradation ignored the geometry distortion in $Z$ axis, which results in reduction of the distance for the redesigned cloak to steer the waves before they arrive at the inner boundary. Hence, the redesigned cloak is an approximate acoustic cloak.

The performances of the ideal acoustic cloak denoted by Equation (1.23) and the approximate acoustic cloak denoted by Equation (1.25) were validated by using commercial software COMSOL Multiphysics’ PDE solver of Helmholtz equation $\nabla \cdot (c \nabla p) + ap = 0$, which has the exactly same form as Equation (1.19).

Assume a harmonic acoustic plane wave $p = e^{-jkx}$ is horizontally incident on the cloak, where $j^2 = -1$, $k = 2\pi/\lambda$ is the wave number with $\lambda$ denotes the corresponding wavelength. Figure 1.16 shows the simulation domain. The background medium is air with $\rho_0 = 1.225\text{ kg/m}^3$ and $K_0 = 1.42 \times 10^5\text{ Pa}$, the material for the medium of the cloak is determined by Equation (1.23) or Equation (1.25). The material parameters for the obstacle within the cloak are $K_{ob} = K_0, \rho_{ob} = \rho_0/4$. In the simulation below, the wavelength of the incident plane wave is given as $\lambda = 0.06\text{ m}$, exterior and interior radii of the cloaks are designed as 0.2 m and 0.1 m respectively.
1.3. Cloak

In COMSOL, Dirichlet boundary condition \( p = e^{-jkx} \) was applied to the left boundary of the simulation domain to mimic the incident plane wave, Neumann boundary conditions \( n \cdot \nabla p = 0 \) were applied to the other three boundaries. Figure 1.17(a) shows the \( p \) field in the simulation domain with the boundary conditions mentioned above when there is no obstacle, obviously plane wave was generated.

Figure 1.17(b) illustrates the \( p \) field of the incident plane wave scattered by the obstacle without a cloak, Figure 1.17(c) shows the \( p \) field of the incident wave with the obstacle surrounded by the designed ideal cloak, it can be observed that the obstacle with the designed ideal cloak don’t disturb the incident wave outside them, namely, the obstacle is “invisible” to the incident acoustic wave. Theoretically, the ideal acoustic cloak has perfect performance at every frequency for its parameters showed in Equation (1.23) are frequency independent.

Figure 1.17(d) shows the performance of the approximate acoustic cloak, it can be observed that the approximate cloak can almost makes the rigid obstacle “invisible” since the scatter of tiny part of the incident wave by the obstacle is unavoidable.

More details of the performances of the ideal cloak and the approximate cloak are illustrated in Figure 1.18.

### 1.3.3 Cloak for elastic waves

The governing equations for the waves \( \mathbf{U}(\mathbf{X}) \) in elastic materials are:

\[
\begin{align*}
\nabla_X \cdot \mathbf{\sigma} &= j\omega \mathbf{P} \\
\mathbf{\sigma} &= \mathbf{C}^0 : \nabla_X \mathbf{U} \\
\mathbf{P} &= j\omega \rho_0 \mathbf{U}
\end{align*}
\]  

(1.27)

here, \( \mathbf{P} \) is the momentum of the particle, \( \mathbf{\sigma} \) is the stress tensor, note that the \( j \) in front of a term is not an index but satisfies \( j^2 = -1 \).

In order to obtain the elastic cloak, except the geometrical transformation \( \mathbf{F} \), the variable transformation will also be designed:

\[
\mathbf{U} = \mathbf{A}^T \mathbf{u}, \quad \mathbf{u}_I = A_{iI} u_i
\]  

(1.28)
Matrix $A$ in the above equation is termed “gauge” in [64].

Implementing the transformation in Equation (1.14) and Equation (1.28) on Equation (1.27), the governing equations for the elastic cloak can be obtained [64]:

$$
\nabla \cdot \sigma = j\omega p
$$

$$
\sigma = C_{\text{eff}} : \nabla u + j\omega S_{\text{eff}} : u
$$

$$
p = S_{\text{eff}} : \nabla u + j\omega \rho_{\text{eff}} : u
$$

with parameters:

$$
C_{\text{eff}}_{ijkl} = J C_{ijKL}^{0} \theta_{ijkl} \theta_{KL}
$$

$$
S_{\text{eff}}_{ijl} = J C_{ijKL}^{0} \theta_{ijkl} \theta_{KL,k}
$$

$$
\rho_{\text{eff}}_{jl} = \rho_{jl} + J C_{ijKL}^{0} \theta_{ijkl} \theta_{KL,k}
$$

Figure 1.17: (a) Pressure field without obstacle. (b) Pressure field with a bare obstacle, circle dash line indicates the obstacle. (c) Pressure field with an obstacle surrounded by an ideal cloak, circle dash lines indicate the boundaries of the cloak. (d) Pressure field with an obstacle surrounded by an approximate cloak, circle dash lines indicate the boundaries of the cloak.
1.3. Cloak

Figure 1.18: Pressures along the central line $y = 0$. Red line, blue dot-dash line and green dot-dash line represent the case without obstacle, the case with an obstacle surrounded by an ideal cloak and the case with an obstacle surrounded by an approximate cloak, respectively. Vertical dot-dash lines represent inner boundaries of the cloaks, vertical dash lines represent outer boundaries of the cloaks.

Here, $\vartheta_{ij} = J^{-1}F_{iI}A_{jJ}$, density $\rho_{jl}$ is component of:

$$\rho = \rho_0 J^{-1}AA^T$$  \hspace{1cm} (1.31)

It can be observed from Equations (1.30) and (1.31) that the parameters for the elastic cloak are determined by the geometry transformation $F$ and the variable transformation $A$. The geometry transformation $F$ is related with the shape of the cloak desired but the variable transformation $A$ is quit free to choose, therefore, different parameters for elastic cloak can be obtained by designating different $A$.

To mimic the elastic cloak indicated by Equation (1.29) only by designing materials, from the view of practice, two types of $A$ are proposed in the literature [64]. The first one designates $A = F$, materials corresponding to these transformations are Willis elastic materials possessing symmetrical elasticity matrix but anisotropic density. The second one makes $A = I$, the corresponding materials are Cosserat elastic materials having isotropic density but asymmetrical strain (stress) tensor leading to a unusual 9-by-9 elasticity matrix. Examples for these two types of cloaks will be given below.

The initial material is assumed to be isotropic and homogeneous, with elasticity matrix in the Voigt notation ($\{1, 2, 3, 4, 5, 6\} = \{11, 22, 33, 23, 31, 12\}$) given by:

$$C^0 = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}$$  \hspace{1cm} (1.32)
Adopting the geometry transformation in Equation (1.22) to create a cylindrical cloak. Assume \( A = F \), the elasticity matrix for the corresponding material is a 6-by-6 matrix, its nonzero components in Voigt notation are:

\[
C_{1111}^{\text{eff}} = (\lambda + 2\mu)\left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}, \quad C_{2222}^{\text{eff}} = (\lambda + 2\mu)\left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r} \frac{r - R_1}{r}
\]

\[
C_{3333}^{\text{eff}} = (\lambda + 2\mu)\left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}, \quad C_{3331}^{\text{eff}} = \mu \frac{r - R_1}{r}
\]

\[
C_{1122}^{\text{eff}} = \lambda\left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}, \quad C_{1113}^{\text{eff}} = C_{1133}^{\text{eff}} = \lambda \frac{r - R_1}{r}
\]

\[
C_{2233}^{\text{eff}} = C_{3332}^{\text{eff}} = \lambda \frac{r - R_1}{r}.
\]

For the \( A = I \) case, the elasticity matrix for the corresponding material is a 9-by-9 matrix, here the 9-index Cosserat notation is used, which means:

\[
\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{11, 22, 33, 23, 32, 31, 13, 12, 21\} \quad (1.34)
\]

The nonzero components in the matrix are:

\[
C_{1111}^{\text{eff}} = (\lambda + 2\mu)\frac{r - R_1}{r}, \quad C_{2222}^{\text{eff}} = (\lambda + 2\mu)\frac{r}{r - R_1}, \quad C_{3333}^{\text{eff}} = (\lambda + 2\mu)\left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}
\]

\[
C_{2323}^{\text{eff}} = \mu \frac{r - R_1}{r}, \quad C_{3232}^{\text{eff}} = \mu \left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}, \quad C_{3131}^{\text{eff}} = \mu \left(\frac{R_2 - R_1}{R_2^2}\right)^2 \frac{r - R_1}{r}
\]

\[
C_{1313}^{\text{eff}} = \mu \frac{r - R_1}{r}, \quad C_{1212}^{\text{eff}} = \mu \frac{r - R_1}{r}, \quad C_{2121}^{\text{eff}} = C_{1122}^{\text{eff}} = \lambda
\]

\[
C_{1133}^{\text{eff}} = C_{3311}^{\text{eff}} = \lambda \frac{R_2 - R_1}{R_2^2} \frac{r - R_1}{r}, \quad C_{2233}^{\text{eff}} = C_{3322}^{\text{eff}} = \lambda \frac{R_2 - R_1}{R_2^2} \frac{r - R_1}{r}, \quad C_{2332}^{\text{eff}} = C_{3232}^{\text{eff}} = \mu \frac{R_2 - R_1}{R_2^2}
\]

\[
C_{3113}^{\text{eff}} = C_{3133}^{\text{eff}} = \mu \frac{R_2 - R_1}{R_2^2} \frac{r - R_1}{r}, \quad C_{1221}^{\text{eff}} = C_{1122}^{\text{eff}} = \mu
\]

(1.35)

1.3.4 Cloak for flexural waves in plates

The fourth-order partial differential equation governing the out-of-plane displacement amplitude \( w(X) \) of an isotropic and homogeneous plate is:

\[
D_0 \nabla_X^4 w - \rho_0 h \omega^2 w = 0 \quad (1.36)
\]

Here, \( D_0 \) is the flexural rigidity.

For the facility of transformation, Equation (1.36) can be rewritten in the form below:

\[
(\nabla_X^2 + \beta^2)(\nabla_X^2 - \beta^2)w = 0, \quad \beta = (\frac{\rho_0 h}{D_0} \omega^2)^{1/4} \quad (1.37)
\]

Adopting the geometry transformation and assuming \( w(x) = w(X) \), plate equation Equation (1.37) is translated to [70]:

\[
[J \nabla \cdot (J^{-1}FF^T \nabla) + \beta^2][J \nabla \cdot (J^{-1}FF^T \nabla) - \beta^2]w = 0 \quad (1.38)
\]
Therefore, Equation (1.38) is the governing equation of the cloak for flexural waves in thin plate.

It is demonstrated in references [69, 70] that ideal cloak for flexural waves in thin plate indicated by Equation (1.38) can’t be obtained by only designing material parameters in the cloak region. To deal with this issue, two different strategies are proposed, the first one only use anisotropic plate to approximately mimic the cloak by guaranteeing that the fourth-order derivative terms in Equation (1.38) are totally fulfilled and making sure that the other lower order derivative terms are approximately mimicked. The second method involves other terms representing membrane forces and in-plane body forces into the anisotropic plate equation to thoroughly fulfil Equation (1.38), ideal cloak can be obtained by this method. Examples will be given below to briefly demonstrate how these two strategies work.

Adopting the transformation denoted by Equation (1.22) in 2D, the transformed Equation (1.38) expressed in cylindrical coordinates is:

\[
D_0\left((\frac{r-R_1}{r})^2\frac{\partial^4 w}{\partial r^4} + \frac{2}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{1}{r^2(r-R_1)^2} \frac{\partial^4 w}{\partial \theta^4}\right) \\
+ \frac{2(r-R_1)}{r^2} \frac{\partial^3 w}{\partial r^3} - \frac{2}{r^2(r-R_1)} \frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{4}{r^2(r-R_1)^2} \frac{\partial^2 w}{\partial \theta^2} \\
+ \frac{1}{r^2(r-R_1)} \frac{\partial w}{\partial r} - \rho_0 \left(\frac{R_2}{R_2-R_1}\right)^4 \left(\frac{r-R_1}{r}\right)^2 \rho h \omega^2 w = 0
\] (1.39)

Note that coefficients in front derivative terms in Equation (1.39) can be normalized in different ways by dividing different functions. The governing equation for orthotropic and inhomogeneous plate with rigidities and Poisson’s ratios varying radially in cylindrical coordinates is:

\[
D_r \frac{\partial^4 w}{\partial r^4} + \frac{2D_{\theta\theta}}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{D_\theta}{r^2} \frac{\partial^4 w}{\partial \theta^4} + \left(\frac{2D_r}{r^2} + \frac{2 \partial D_r}{\partial r}\right) \frac{\partial^3 w}{\partial r^3} \\
- \left(\frac{2D_{\theta\theta}}{r^2} - \frac{\partial D_{\theta\theta}}{\partial r}\right) \frac{\partial^3 w}{\partial r \partial \theta^2} - \left[\frac{D_r}{r^2} - \frac{\partial D_r}{\partial r} - \frac{1}{r} \frac{\partial (D_r v_\theta)}{\partial r} - \frac{\partial^2 D_r}{\partial r^2}\right] \frac{\partial^2 w}{\partial r^2} \\
+ \left[\frac{2(D_{\theta\theta}+D_\theta)}{r^4} - \frac{1}{r^3} \left(\frac{\partial^2 D_{\theta\theta}}{\partial r^2} + \frac{\partial D_{\theta\theta}}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 (D_r v_\theta)}{\partial r \partial \theta}\right] \frac{\partial^2 w}{\partial \theta^2} \\
+ \left[\frac{D_r}{r^2} - \frac{1}{r} \frac{\partial D_r}{\partial r} + \frac{1}{r} \frac{\partial^2 (D_r v_\theta)}{\partial \theta^2}\right] \frac{\partial w}{\partial r} - \rho h \omega^2 w = 0
\] (1.40)

According to the match of the fourth-order derivative terms in Equation (1.39) and Equation (1.40), flexural rigidities, density and Poisson’s ratio for the approximate cloak can be determined as [69]:

\[
D_r = D_0 (\frac{r-R_1}{r})^2, \quad D_\theta = D_0 (\frac{r-R_1}{r})^2, \quad D_r, \theta = D_0 \\
v_r = v_0, \quad v_\theta = v_0 (\frac{r-R_1}{r})^4, \quad \rho = \rho_0 (\frac{R_2}{R_2-R_1})^4 (\frac{r-R_1}{r})^2
\] (1.41)

Actually, the Poisson’s ratio can be chosen freely as long as it guarantees the approximation. With this parameters and under the condition in Equation (1.42), the difference between Equation (1.39) and Equation (1.40) is very small, hence
Equation (1.39) can be approximately replaced by Equation (1.40). This approximate cloak can manipulate most of the waves to bypass the obstacle before arriving at the inner boundary, its effect was demonstrated in [64].

\[ \frac{R_1}{R_2} \ll 1, \quad \frac{R_1 R_2}{R_2 - R_1} \beta \ll 1 \]  

(1.42)

As aforementioned, to obtain ideal cloak, membrane force \( \mathbf{T} \) and in-plane body force \( \mathbf{S} \) showed below should be applied on the mid-plane of the plate:

\[ \mathbf{T} = \begin{bmatrix} T_{rr} & T_{r\theta} \\ \text{sym} & T_{\theta\theta} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_r \\ S_\theta \end{bmatrix} \]  

(1.43)

These forces are constrained to satisfy the in-plane equilibrium equation:

\[ \nabla \cdot \mathbf{T} + \mathbf{S} = 0 \]  

(1.44)

Governing equations for anisotropic and inhomogeneous plate subjected to these forces are [71]:

\[ \nabla \cdot (\nabla \cdot \mathbf{M}) + \mathbf{T} \cdot \nabla w - \mathbf{S} \cdot \nabla w + \rho h \omega^2 w = 0 \]  

(1.45)

Here, the components of moment tensor are:

\[ \mathbf{M} = \begin{bmatrix} M_{ii} & M_{ij} \\ \text{sym} & M_{jj} \end{bmatrix} \]  

(1.46)

Particularly, Equation (1.45) expressed in cylindrical coordinates is:

\[ \begin{aligned}
D_r \frac{\partial^4 w}{\partial r^4} &+ \frac{2D_{rr}}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{D_\theta}{r^2} \frac{\partial^4 w}{\partial \theta^4} \\
+ \left( \frac{2D_{r\theta}}{r^2} + \frac{2}{r} \frac{\partial D_{r\theta}}{\partial r} \right) \frac{\partial^3 w}{\partial r \partial \theta^2} &- \left( \frac{2D_{r\theta}}{r^2} - \frac{2}{r} \frac{\partial D_{r\theta}}{\partial r} \right) \frac{\partial^3 w}{\partial \theta^2 \partial r} \\
- \left[ \frac{D_\theta}{r^2} - \frac{2}{r} \frac{\partial D_\theta}{\partial r} - \frac{1}{r} \frac{\partial (D_{r\theta}v_\theta)}{\partial r} - \frac{\partial^2 D_r}{\partial \theta^2} - T_{rr} \right] \frac{\partial^2 w}{\partial \theta \partial r} \\
+ \frac{1}{r} \frac{2(D_{r\theta} + D_{rr})}{r^2} &- \frac{1}{r^2} \left( \frac{2D_{r\theta}}{\partial r^2} + \frac{\partial D_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (D_{r\theta}v_\theta)}{\partial r^2} + \frac{T_{\theta\theta}}{r^2} \right] \frac{\partial^2 w}{\partial \theta \partial r} \\
+ \frac{2T_{\theta\theta}}{r^2} \frac{\partial^2 w}{\partial r \partial \theta} &+ \left[ \frac{D_\theta}{r^2} - \frac{1}{r} \frac{\partial D_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 (D_{r\theta}v_\theta)}{\partial r^2} - S_r \right] \frac{\partial w}{\partial r} \\
+ \left( \frac{T_{\theta\theta} - S_\theta}{r^2} \right) \frac{\partial w}{\partial \theta} - \rho h \omega^2 w = 0
\end{aligned} \]  

(1.47)

with the moments:

\[ M_r = -D_r \left( \frac{\partial^2 w}{\partial r^2} + v_\theta \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta \partial r} \right) \right) \]

\[ M_\theta = -D_\theta \left( \frac{1}{r} \frac{\partial w}{\partial r} + v_r \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta \partial r} \right) \]  

\[ M_{r\theta} = -(D_{r\theta} - D_r v_\theta) \left( \frac{1}{r} \frac{\partial w}{\partial r} \right) \]  

(1.48)
From Equation (1.47) it can be observed that the coefficients in front of the third- and fourth-order derivative terms are still only dominated by the flexural rigidities but the coefficients in front of the first- and second-order derivative terms are not only determined by the rigidities but also can be modified by choosing different forces.

To obtain an ideal cloak, the transformed Equation (1.39) need to be renormalized in another way, the principle for the normalization is to guarantee the third- and fourth-order derivative terms in Equation (1.47) are equal with the renormalized equation. Under this principle, the transformed equation (namely Equation (1.39)) is normalized into following form:

\[
D_0 \left[ \frac{r - R_1}{r} \frac{\partial^4 w}{\partial r^4} + \frac{2}{r(r-R_1)} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{1}{r(r-R_1)^3} \frac{\partial^4 w}{\partial \theta^4} \right.
\]

\[
+ \frac{2}{r(r-R_1)^2} \frac{\partial^4 w}{\partial r \partial \theta^3} - \frac{1}{r(r-R_1)} \frac{\partial^2 w}{\partial r^2} + \frac{4}{r(r-R_1)^3} \frac{\partial^2 w}{\partial \theta^2} \left. \right] - \rho_0 \left( \frac{R_2}{R_2-R_1} \right)^4 \frac{r-R_1}{r} \omega^2 w = 0
\]  

The equivalence of all the terms in Equation (1.47) and Equation (1.49) results in the following parameters:

\[
D_r = D_0 \frac{r - R_1}{r}, \quad D_\theta = D_0 \left( \frac{r}{r-R_1} \right)^3, \quad D_{r\theta} = D_0 \frac{r}{r-R_1}, \quad \nu_\theta = \nu_r^{-1} = \left( \frac{r}{r-R_1} \right)^2, \quad \rho = \rho_0 \left( \frac{R_2}{R_2-R_1} \right)^4 \frac{r-R_1}{r}
\]

\[
T_{rr} = D_0 \frac{R_1(3r-2R_1)}{r(r-R_1)^3}, \quad T_{r\theta} = -D_0 \frac{3rR_1}{(r-R_1)^2}, \quad T_{r\theta} = 0
\]

\[
S_r = D_0 \frac{3R_1(2r-R_1)}{r(r-R_1)^3}, \quad S_\theta = 0
\]  

Hence, Equation (1.47) with the parameters in Equation (1.50) indicate an ideal cloak for flexural waves in thin plate.

1.3.5 Remarks

Cloaking theories for acoustic wave, elastic wave and flexural wave in thin plate are introduced in this section. Facilitated by the invariance of the acoustic equation under transformation, acoustic cloak can be obtained by designing the materials inside the cloak and can be practically realized by using acoustic metamaterials [58, 60, 73, 74, 75]. For the elastic equation and the equation for flexural wave in thin plate, they will change the forms under transformation. The realization of elastic cloak requires Willis elastic materials or Cosserat elastic materials. An approximate elastic cloak was proposed by Norris and Parnell [76] by application of finite pre-strain, however its cloaked region is limited. Ideal cloak for flexural wave in thin plate requires anisotropic materials and membrane forces as well as in-plane body forces applied on the middle plane. The approximate cloak for flexural wave
in thin plate can be obtained by only designing materials, but its performance is limited within narrow frequency range with small cloaked region.

1.4 Time-space modulated media

Time-space modulated media are more special GRIN media, their properties are changed both in time and space. The variations in time and space could be independent with each other [77, 78] or coupled with each other [79]. In the coupled case, the modulation acts like a traveling wave in the medium. More attention was payed to this case in the past. Also in this thesis, time-space modulated media specially refer to the coupled cases.

To understand the concept of time-space modulation, Figure 1.19 shows the time-space modulated mass-spring system studied in [80]. The time-space modulation is realized by changing the stiffness of the springs according to sinusoidal wave equation. Consequently, this modulation propagates in time with the velocity \( \pm V \).

![Figure 1.19: One-dimensional harmonic mass-spring system with spatial and temporal sinusoidal modulation of the spring stiffness: \( \beta(x,t) \) as a realization of an elastic time-dependent superlattice. The spatial modulation propagates in time with the velocity \( \pm V \) [80].](image)

Time-space modulated media were firstly exploited in implementations of electromagnetism. A. Cullen [81] proposed a transmission line with distributed time-space varying inductances to obtain parametric amplification. Later, J. C. Simon [82] studied the influence of the time-space periodic modulation on the electromagnetic waves by using perturbation method. He discussed a frequency conversion phenomenon induced by time-space modulated media. J. C. Simon’s study by only retaining the fundamental and first orders harmonics was questioned by A. Hessel and A. Oliner [82, 83], the latter developed a rigorous solution for propagation of waves in time-space modulated media by expressing the wave in a Floquet form and taking into account all of the harmonics. Using this rigorous method, E. Cassedy and A. Oliner [82, 83, 79, 84] studied the interactions of electromagnetic waves with the time-space modulation, and discussed the stability of the time-space varying system. Because of the time-varying feature, time-space modulated media essentially break the time-reversal symmetry, consequently break the reciprocity theorem constraining the physical behavior of media supporting wave propagation [85]. Thus,
in recent years, time-space modulated media were exploited to realize nonreciprocal wave propagation. H. Lira et al. [86] used the time-space modulation to stimulate mode conversion from one direction but leave the mode from the opposite direction unchanged to realize nonreciprocal electromagnetic wave transmission. D. Wang et al. [87] built an optical diode by periodically modulating the refractive index of photonic crystals both in time and space.

Time-space modulated media were also intensively studied in vibrations and acoustics in recent two decades. Ordinary (static) materials, like metals and traditional composites, are all passive and possess conventional properties, which are difficult to meet the requirements for designing advanced structures. To overcome these limitations, A. Lurie [88] proposed to vary the properties of materials in a suitable spatio-temporal pattern, to make the materials be able to respond adequately at each time to the demand required by the environment. These smart materials are termed "dynamic materials" by A. Lurie, as well as his collaborators and his followers. In [89], A. Lurie designed an array composed of cells with two segments, the properties of these segments are changing in time and space, like the whole array is moving with a constant velocity. He obtained the effective material parameters of the dynamic array and predicted that by appropriately designing the activated array, unidirectional wave propagation can be observed. Following A. Lurie’s work, S. Weeles [90] validated his homogenization results by performing direct numerical simulations. J. Jensen [91] optimized a dynamical rod to minimize the transmitted energy when a wave package is incident on it. He also found that the frequency of the transmitted wave is split, i.e., except the frequency component corresponding to the incident wave, higher order components were also seen. L. Shui et al. [92] used an improved multi-scale homogenization method to obtain the effective material parameters of the dynamic material. According to the homogenization results, they predicted asymmetric wave propagation in the dynamic material and numerically demonstrated it. Different from A. Lurie and his followers' works, recently elastic wave propagation in time-space modulated structures was studied by exploiting the virtue of periodicity. Dispersion relations of Bloch modes supported by the modulated structures were obtained using the Bloch theory [93, 94]. It was found that the band structures are asymmetric, i.e., the stop bands of the fundamental Bloch modes are separated. Also nonreciprocal elastic wave propagation was observed within these bands.

In brief, Being extensively studied in diverse areas, many unusual phenomena induced by time-space modulation have been observed: nonreciprocal (or say unidirectional) wave propagation [89, 90, 92, 80, 93, 94, 95], frequency splitting [91, 95], frequency conversion [82, 95, 96, 97] and parametric amplification [81, 82].

1.4.1 Nonreciprocal wave propagation

The most attractive property of the time-space modulated media is the nonreciprocity. In conventional propagation media, wave motion obeys a fundamental property: reciprocity, which describes the symmetry in wave transmission between two points
in space. Reciprocity guarantees that wave propagation always occurs in a symmetrical fashion. If waves can make their way from a source to an observer, the opposite propagation path, from the observer to the source, is equally possible and the transmission is symmetric. For example, in the medium shown in Figure 1.20, a source $S$ and a detector $D$ are respectively placed at two different points. The reciprocity states that the signal measured by the detector will remain the same when the locations of the source and the detector are exchanged with each other.

The nonreciprocity means that the above introduced properties is violated. The nonreciprocity of time-space modulated media is directly indicated by the symmetry-broken illustrated by their band structures. Figure 1.21 shows the dispersion curves of longitudinal Bloch modes in a beam with time-space modulated Young's modulus obtained in [93]. It is obvious that this band structure is asymmetrical, i.e., if $(k, \omega)$ represent a positive-going Bloch mode in the modulated structure, there isn’t a corresponding negative-going Bloch modes with $(-k, \omega)$. Also, the two stop bands occupy totally different frequency ranges. These bands are called directional band gaps [93, 94], namely, waves can only propagate in one direction within these bands. We remark here that this conclusion is not appropriate, more details about this remark and properties of the band structures of time-space modulated media will be discussed in Chapter 5.

The nonreciprocal propagation in time-space modulated media was numerically demonstrated in [93, 94]. As shown in Figure 1.22, it is verified that, within the stop bands (see Figure 1.21), most of the generated waves inside a time-space modulated beam will travel to one direction. The strong nonreciprocity shown in Figure 1.22 maybe exploited to realized one-way energy insulators [93, 94]. However, as will be discussed in Chapter 6, this one-way energy insulation will fail in finite systems due to the frequency conversion caused by the time-space modulation.

1.4.2 Other special phenomena induced by time-space modulated media

The frequency splitting and frequency conversion are also two interesting phenomena induced by time-space modulated media. Frequency splitting means that, a
Figure 1.21: Band structure of the longitudinal modes in a beam with time-space modulated Young’s modulus [93]. $\mu$ and $\Omega$ are dimensionless wavenumber and frequency, respectively.

single harmonic incident on the time-space modulated medium will generate many harmonics with different frequencies. These nonlinear phenomenon is numerically observed in [91, 95, 97]. Frequency conversion can be treated as a special case of frequency splitting. It means that the main frequency of a wave group, after interacting with the time-space modulated media, is down or up converted by a value equal to the modulation frequency. This phenomenon was first theoretically introduced by J. Simon [82] in terms of electromagnetism waves, then numerically demonstrated in [95, 96, 97] very recently. In Chapters 5 and 6, these unusual phenomena are further theoretically studied and the mechanisms behind them are explained.

1.5 Conclusions

Controlling waves using GRIN media is a long studied area but still growing very fast nowadays. Most of the studies in this area originated from electromagnetism, afterwards being expanded and developed in other domains. Researches in this area link knowledges from different disciplines, such as physics, material science, acoustics and mechanics, to name a few. This feature promotes collaborations between researchers from those diverse disciplines. It also bridges the gap between pure physical theories and applications.

In terms of the acoustic and elastic waves, during the last several decades GRIN media were successfully exploited to focus and trap waves, even to cloak objects from waves, to obtain nonreciprocal propagation. These fascinating phenomena
Figure 1.22: (a) Schematic of a $2L$ long time-space modulated beam loaded in its mid-span, longitudinal motion is excited by the horizontal force $F_{\text{ext},L}$. (b) Forced wave propagation in the beam in Figure 1.22(a) when the frequency of the excitation is within the left stop band in Figure 1.21 (left) and when the frequency of the excitation is within the right stop band in Figure 1.21 (right). These results are from [93].

broaden the way we control waves in acoustics and mechanics. They already found applications or may find in SHM, energy harvesting, acoustic/vibration control, etc.

To date, GRIN media were mainly engineered by using artificial passive PCs and metamaterials or through tailoring thickness of beams and plates. In our studies, we explore to introduce piezoelectric materials into GRIN media for the purpose of designing adaptive GRIN structures to control guided waves. The properties of piezoelectric materials and they prospect of being used to build GRIN structures are fully discussed in the next chapter.
Chapter 2

Tunable ability of shunted piezoelectrics

Contents

2.1 Overview of piezoelectric materials ........................................ 29
2.2 Introduction to negative capacitance ........................................... 31
2.3 Effective material parameters of piezoelectrics shunted with negative capacitance ................................................................. 33
2.4 Dynamical properties of unit cells containing piezoelectrics shunted with negative capacitance ....................................................... 35
   2.4.1 Effective medium model ......................................................... 35
   2.4.2 Discussion ............................................................................ 38
2.5 Recent advances in wave control using piezoelectric materials ........ 39
   2.5.1 Adaptive metamaterials ......................................................... 39
   2.5.2 Hybrid dispersive media ......................................................... 40
   2.5.3 Reconfigurable structures ....................................................... 40
   2.5.4 Gradient index interfaces ...................................................... 41
2.6 Conclusions .............................................................................. 42

In this chapter, tunable ability of piezoelectric materials particularly shunted with Negative Capacitance (NC) is fully discussed. Overview of piezoelectric materials is made in Section 2.1. Practical realization of NC values is introduced in Section 2.2. After that, effective material parameters of piezoelectrics shunted with NC are obtained. Then, effective bending stiffness and density of a unit cell containing the shunted piezoelectrics are derived by using an Effective Medium Model (EMM) in Section 2.4, accuracy of the EMM is also discussed by comparing with the Finite Element Method (FEM). Recent advances in wave control using piezoelectric materials is covered in Section 2.5. Some conclusions are summarized in Section 2.6.

2.1 Overview of piezoelectric materials

Piezoelectric materials belong to the so-called smart materials, or multi-functional materials, which have the ability to respond significantly to stimuli of different physical natures. The ability of piezoelectric materials is the piezoelectric effect, which was discovered by Pierre and Jacques Curie in 1880. The direct piezoelectric
effect permit the piezoelectric material to generate an electrical charge in proportion to an externally applied force. On the other hand, the inverse piezoelectric effect induces an expansion of the material when an electric field parallel to the direction of polarization is applied.

The most popular piezoelectric materials are Lead-Zirconate-Titanate (PZT) which is a ceramic, and Polyvinylidene fluoride (PVDF) which is a polymer [98]. These materials are artificially made through a polling process. Nature materials that exhibit piezoelectric effects have randomly orientated built-in electric dipoles below a certain temperature called the Curie temperature. These materials show no piezoelectricity since the net electric dipole on a macroscopic scale is zero. To form the piezoelectricity, the material need to be heated above the Curie temperature and then be cooled down in the presence of a high electric field. During this polling process, the inner dipoles tend to align, leading to an electric dipole on a macroscopic scale along the polling field. After cooling and removing the applied field, the dipoles remains aligned along the polling direction thus the material becomes permanently piezoelectric. Note that the piezoelectricity will lose if the PZT or PVDF work at a temperature higher than the Curie one or they are subjected to excessive electric fields in the direction opposed to the polling direction.

The PZT materials are used as transducers in our studies for they are more efficient [98]. A typical piezoelectric transducer made by PZT is shown in Figure 2.1. Assume that directions 1, 2 and 3 correspond to \( x \), \( y \) and \( z \) axes, respectively, as shown in Figure 2.1. The piezoelectric effect of this material can be expressed by the following constitutive equations:

\[
S = d^T E + S^E T \\
D = \varepsilon^T E + d T
\]  

(2.1)

in which, \((\cdot)^T\) indicates matrix transpose, \(D\) and \(E\) are respectively vectors of
2.2 Introduction to negative capacitance

electrical displacements and electrical fields in the material, \( S \) and \( T \) are material engineering strains and material stresses, respectively. Components of them are given below:

\[
D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix}
\]

For the constitutive parameters in Equations (2.1), \( \varepsilon^\sigma \) is the dielectric constants matrix measured at constant stress, \( d \) is the piezoelectric coupling constants matrix, \( S^E \) is the compliance matrix measured at constant electric field. Details of them are given below:

\[
\varepsilon^\sigma = \begin{bmatrix} \varepsilon^\sigma_1 \\ 0 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{31} & d_{32} & d_{33} & 0 & 0 \end{bmatrix}
\]

and:

\[
S^E = \begin{bmatrix}
S^E_{11} & S^E_{12} & S^E_{13} & 0 & 0 & 0 \\
S^E_{12} & S^E_{22} & S^E_{23} & 0 & 0 & 0 \\
S^E_{13} & S^E_{23} & S^E_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S^E_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S^E_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S^E_{66} 
\end{bmatrix}
\]

PZT materials are usually transverse isotropic [99], which means \( \varepsilon^\sigma_1 = \varepsilon^\sigma_2 \), \( d_{31} = d_{32} \), \( d_{24} = d_{15} \), \( S^E_{11} = S^E_{22} \) and \( S^E_{13} = S^E_{23} \).

According to the constitutive equations in Equation (2.1), the piezoelectric material commonly has three different actuation modes, as shown in Figure 2.2. In the \( d_{33} \) mode, an electric field \( E_3 \) applied along the polarization direction will cause an extension of the material along the same direction. Also, a shrinkage is observed along the direction 1 and 2 dominated by the coefficients \( d_{31} \) and \( d_{32} \), which are equal and negative. The \( d_{31} \) mode is often used when a thin piezoelectric patch is bonded on the surface of plate-like structures. In this mode, the expansion and contraction of the supporting structure cause the deformation of the piezoelectric patch along direction 1 and 2, consequently generating an electric field along direction 3. In the \( d_{15} \) mode, an electric field \( E_1 \) or \( E_2 \) will produce a shear deformation \( S_{13} \) or \( S_{23} \) due to the coefficient \( d_{15} \) or \( d_{24} \).

2.2 Introduction to negative capacitance

In practice, the negative capacitance can be realized by a synthetic circuit as used in F. Tateo et al.’s works [100, 101]. The layout of this circuit is described in
Figure 2.2: Actuation modes of piezoelectric actuators, $P$ indicates the direction [98].

Figure 2.3. It contains a number of passive components, four resistors $R_1, R_2, R_3, R_4$ and a capacitor $C$, as well as an operational amplifier (Op-Amp). The equivalent impedance of this circuit is:

$$Z_{eq}(\omega) = R_1 - \frac{R_3}{R_4} \frac{1}{j\omega C + \frac{1}{R_2}}$$

(2.2)

Here, $\sqrt{-1} = -1$. According to F. Tateo et al., the formula in Equation 2.2 can be simplified without loss of generality. First, the resistor $R_2$, which is necessary for the stability of the Op-Amp at DC, is sufficiently large to be considered negligible. Second, the resistor $R_1$, is required in practical realization of the circuit, can be selected small enough to be negligible. Therefore, the equivalent impedance in Equation 2.2 can be simplified as:

$$Z_{eq}(\omega) = -\frac{R_3}{R_4} \frac{1}{j\omega C + \frac{1}{R_2}}$$

(2.3)

Thus, a required negative capacitance value $C_{neg} = -\frac{R_4 C}{R_3}$ can be obtained by tuning.
2.3 Effective material parameters of piezoelectrics shunted with negative capacitance

In this section, tunable feature of the piezoelectric patch in Figure 2.1 shunted with NC is studied. The effective compliance matrix of the shunted piezoelectrics is obtained as functions of the NC value $C_{neg}$. The polling direction of the piezoelectric patch is direction 3. Electrodes cover the upper and lower surfaces of the patch. Assume that electric field within and electrical displacement on the electrodes are uniform for the piezoelectric patch [2]. Therefore, a variable change can be performed according to relations below:

\[
V = -LE \\
Q = -AD
\]  

(2.4)

in which:

\[
L = \begin{bmatrix} l_p & 0 & 0 \\ 0 & l_p & 0 \\ 0 & 0 & h_p \end{bmatrix}, \quad A = \begin{bmatrix} l_p h_p & 0 & 0 \\ 0 & l_p h_p & 0 \\ 0 & 0 & l_p^2 \end{bmatrix}, \quad V = \begin{bmatrix} 0 \\ 0 \\ V \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix}
\]

here, $V$ and $Q$ are vectors of voltages on surfaces of the piezoelectric patch and charges flowing into these surfaces, respectively. The shunting circuit is connected to the electrodes perpendicular to the direction 3 (namely the $z$ axis), thus only the third elements in vectors $V$ and $Q$ are nonzero. $l_p$ and $h_p$ are respectively length and thickness of the piezoelectric patch as shown in Figure 2.1.

We define the output current from the piezoelectric patch as $I = -\dot{Q}$. Therefore the voltage $V$ and the charges $Q$ are related through [102]:

\[
Q = -C_{neg}V
\]  

(2.5)
Substituting Equations (2.4) and (2.5) into Equation (2.1), we can find a new expression for the strain only in terms of stress:

\[
S = [S^E - d^T L^{-1}(C_{neg} + C_P^T)^{-1}A]T
\]

(2.6)

in which:

\[
C_{neg} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & C_{neg}
\end{bmatrix}, \quad C_P^T = A \varepsilon^p L^{-1}
\]

\(C_P^T\) is a diagonal matrix, its diagonal term \(C_{Pi}\) represents the intrinsic capacitance between the surfaces perpendicular to the direction \(i\) at constant stress.

According to Equation (2.6), the effective compliance matrix of the shunted piezoelectric patch is expressed as:

\[
S_{eff} = S^E - d^T L^{-1}(C_{neg} + C_P^T)^{-1}A
\]

(2.7)

The effective compliance matrix is symmetrical, it has the following form:

\[
S_{eff} = \begin{bmatrix}
S_{11}^{eff} & S_{12}^{eff} & S_{13}^{eff} & 0 & 0 & 0 \\
S_{12}^{eff} & S_{22}^{eff} & S_{23}^{eff} & 0 & 0 & 0 \\
S_{13}^{eff} & S_{23}^{eff} & S_{33}^{eff} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44}^{eff} & 0 \\
0 & 0 & 0 & 0 & S_{55}^{eff} \\
0 & 0 & 0 & 0 & 0 & S_{66}^{eff}
\end{bmatrix}
\]

(2.8)

Components in the matrix are:

\[
S_{11}^{eff} = S_{22}^{eff} = S_1^E - \frac{d_{31}^2 l_p^2}{h_p(C_{P3} + C_{neg})}, \quad S_{33}^{eff} = S_{33}^E - \frac{d_{33}^2 l_p^2}{h_p(C_{P3} + C_{neg})}
\]

\[
S_{44}^{eff} = S_{44}^E - \frac{d_{13}^2 h_p}{C_{P2}}, \quad S_{55}^{eff} = S_{55}^E - \frac{d_{15}^2 h_p}{C_{P1}}, \quad S_{66}^{eff} = S_{66}^E
\]

(2.9)

\[
S_{12}^{eff} = S_{12}^E - \frac{d_{31}^2 l_p^2}{h_p(C_{P3} + C_{neg})}, \quad S_{13}^{eff} = S_{23}^{eff} = S_{13}^E - \frac{d_{31} d_{33} l_p^2}{h_p(C_{P3} + C_{neg})}
\]

According to the expressions of the components in \(S_{eff}\), we can see that the shunted piezoelectric patch is still transverse isotropic just as the short-circuit one. Accordingly, like other transverse isotropic materials [103], we can define 5 independent parameters to describe the material properties of the shunted piezoelectric patch:

\[
E_1^{eff} = \frac{1}{S_{11}^{eff}}, \quad E_3^{eff} = \frac{1}{S_{33}^{eff}}, \quad G_{13}^{eff} = \frac{1}{S_{13}^{eff}}
\]

\[
\nu_{12}^{eff} = -\frac{S_{12}^{eff}}{S_{11}^{eff}}, \quad \nu_{13}^{eff} = -\frac{S_{13}^{eff}}{S_{11}^{eff}}
\]

(2.10)
2.4. Dynamical properties of unit cells containing piezoelectrics shunted with negative capacitance

in which, $E_{1}^{\text{eff}}$ is the Young’s modulus along the in-plane directions (i.e., directions 1 and 2), $E_{3}^{\text{eff}}$ is the transverse Young’s modulus along direction 3, $G_{13}^{\text{eff}}$ is the shear modulus between the in-plane and transverse directions, $\nu_{12}^{\text{eff}}$ and $\nu_{13}^{\text{eff}}$ are Poisson’s ratios linking deformations of corresponding directions. Using these parameters, $S^{\text{eff}}$ can also be represented in the form:

\[
S^{\text{eff}} = \begin{bmatrix}
\frac{1}{E_{1}^{\text{eff}}} & -\frac{\nu_{12}^{\text{eff}}}{E_{1}^{\text{eff}}} & -\frac{\nu_{13}^{\text{eff}}}{E_{3}^{\text{eff}}} & 0 & 0 & 0 \\
-\frac{\nu_{12}^{\text{eff}}}{E_{1}^{\text{eff}}} & \frac{1}{E_{1}^{\text{eff}}} & -\frac{\nu_{13}^{\text{eff}}}{E_{3}^{\text{eff}}} & 0 & 0 & 0 \\
-\frac{\nu_{13}^{\text{eff}}}{E_{3}^{\text{eff}}} & -\frac{\nu_{13}^{\text{eff}}}{E_{3}^{\text{eff}}} & \frac{1}{E_{3}^{\text{eff}}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{13}^{\text{eff}}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{13}^{\text{eff}}} & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + \nu_{12}^{\text{eff}})\frac{1}{E_{1}^{\text{eff}}}
\end{bmatrix}
\]

(2.11)

From Equations (2.9) and (2.10) we can see that, the NC circuit have influence on the effective Young’s moduli and Poisson’s ratios. Figure 2.4 shows the variation of these parameters with respect to the $C_{\text{neg}}$. In Figure 2.4(a), $E_{1}^{\text{sh}} = 1/S_{11}^{E}$ is the in-plane Young’s modulus of the short-circuit piezoelectric material. It can be seen that the extent of the tuning range of these parameters are very large. Theoretically they can be varied from $-\infty$ to $+\infty$. However in practice, this is not available. To have admissible materials which are thermodynamically stable, the Young’s moduli must be positive and also the Poisson’s ratios are restricted by certain constraints [103]. The stability issues of the shunted piezoelectric patches are not discussed here, because these patches will be bonded with other passive structures, it is meaningful only when the stability of the whole system is discussed.

2.4 Dynamical properties of unit cells containing piezoelectrics shunted with negative capacitance

As demonstrated above, the shunting NC circuit can significantly change the material parameters of the piezoelectric patches. Therefore, when these shunted patches are used in the cell shown in Figure 2.5, dynamical properties of the cell can be tuned by varying the shunting $C_{\text{neg}}$. To study this tunable feature, we developed an EMM for the cell and discussed the accuracy of it.

2.4.1 Effective medium model

Our interest is to control flexural waves in plates (i.e., the $A_{0}$ mode), which carries most of the energy in plates [5]. The EMM for flexural waves is obtained in three steps [104]. The first step is to represent the effective material parameters of the
Chapter 2. Tunable ability of shunted piezoelectrics

Figure 2.4: Variation of the (a) effective Young’s moduli and (b) effective Poisson’s ratios with respect to the $C_{neg}$.

shunted piezoelectric patches as functions of the negative capacitance value, which has already been done in Section 2.1. The thickness of the piezoelectric patches are much smaller than the host plate’s thickness, hence, it is reasonable to assume that all piezoelectric patches are under plane stress condition when the plate is experiencing flexural motion, i.e., $T_3 = T_4 = T_5 = 0$. Therefore, the constitutive relations of the shunted piezoelectric patch are reduced to:

$$
\begin{bmatrix}
S_1 \\
S_2 \\
S_6
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E^{eff}_1} & \frac{\nu^{eff}_{12}}{E^{eff}_1} & 0 \\
\frac{\nu^{eff}_{12}}{E^{eff}_1} & \frac{1}{E^{eff}_1} & 0 \\
0 & 0 & \frac{2(1 + \nu^{eff}_{12})}{E^{eff}_1}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_6
\end{bmatrix}
$$

(2.12)

In what follows, for the simplicity, the two parameters $E^{eff}_1$ and $\nu^{eff}_{12}$ in Equation (2.12) are represented as $E_p$ and $\nu_p$, respectively. According to Equations (2.9) and (2.10), the detailed expressions of these two parameters are:
2.4. Dynamical properties of unit cells containing piezoelectrics shunted with negative capacitance

Figure 2.5: Top and side views of the elementary cell containing piezoelectric patches shunted with NC circuits.

\[ E_p = E_p^{sh} \frac{C_{neg} + C_{p3}^T}{C_{neg} + C_{p3}(1 - k_{31}^2)} \]  
\[ \nu_p = \nu_p^{sh} \frac{C_{neg} + C_{p3}^T(1 + k_{31}^2/\nu_p^{sh})}{C_{neg} + C_{p3}^T(1 - k_{31}^2)} \]  

(2.13)

Here, \( \nu_p^{sh} = -S_{12}^{E}/S_{11}^{E} \) is the in-plane Poisson’s ratio of the short-circuit piezoelectric material; \( k_{31} = d_{31}/\sqrt{S_{11}^{E}/\varepsilon_0} \) is the coupling factor.

The second step is to determine the effective parameters of the shunted piezoelectric sandwich structure highlighted by the dashed lines in Figure 2.5. According to the classical laminated plate theory, the effective bending stiffness of the piezoelectric sandwich structure can be expressed as:

\[ \rho_A = \rho_{b} h_{b} + 2 \rho_{p} h_{p} \]
\[ D_A = D_{b} + \frac{2E_p}{3(1-\nu_p^2)}[(\frac{h_b}{2} + h_p)^3 - (\frac{h_b}{2})^3] \]

(2.14)

Here, \( D_b = \frac{E_b h_b^3}{12(1-\nu_b^2)} \) is the bending stiffness of the host plate and \( E_b, \nu_b \) denote the Young’s modulus and the Poisson’s ratio of the host plate respectively; \( \rho_b \) and \( \rho_p \) are the densities of the host plate and the piezoelectric patches respectively.

The last step is to derive the effective parameters of the entire cell. With Equations (2.13) and (2.14), the effective area density and the effective bending stiffness of the entire unit cell can be obtained according to the rule and inverse rule of mixtures:

\[ \rho_{eff} = \chi \rho_{A} + (1 - \chi) \rho_{b} h_{b} \]
\[ D_{eff} = \frac{D_{A} D_{b}}{\chi D_{A} + (1 - \chi) D_{b}} \]  

(2.15)
Chapter 2. Tunable ability of shunted piezoelectrics

Figure 2.6: Variation of the effective bending stiffness with the $C_{neg}$. Red stars indicate the $C_{neg}$ values used in the simulations in Section 2.4.2.

here, $\chi = (l_p/l_b)^2$ is the ratio of the surface area covered by the piezoelectric patch to the surface area of the unit cell.

According to Equations (2.13), (2.14) and (2.15), it can be seen that the NC circuit has no effect on the effective area density but will change the effective bending stiffness. Figure 2.6 shows the variation of the effective bending stiffness with respect to the shunting $C_{neg}$. The effective bending stiffness must be positive to avoid creating power into the system, namely to have a stable system. The $C_{neg}$ region corresponding to the unstable zone is highlighted in the figure. Outside the unstable zone, it is obvious that we can soften or stiffen the cell within a large range by changing $C_{neg}$.

2.4.2 Discussion

The accuracy of the EMM is discussed in this section. We used the EMM to study the dispersion curves of the flexural mode ($A_0$ mode) in structures composed of identical cells shown in Figure 2.5. These curves are easily obtained by the dispersion relation:

$$\omega = k^2 \sqrt{\frac{D_{eff}}{\rho_{eff}}}$$

in which, $k$ is the module of the wave vector.

We compared these dispersion curves with those obtained by using the FEM and Bloch boundary Conditions (B.C) (see Appendix B). We used 6 different $C_{neg}$ values in our simulations, 3 in the soften zone and 3 in the stiffen zone, they are indicated in Figure 2.6 by the red stars. In the simulations, the geometry parameters of the unit cell are: $l_b = 0.04 m$, $l_p = 0.035 m$, $h_b = 0.005 m$ and $h_p = 0.001 m$. Material parameters of the host plate and piezoelectric patches are listed in Appendix A.

Figure 2.7 shows the dispersion relations at edges of the Irreducible Brillouin Zone (namely, the zone surrounded by OMΓ shown in the figure) corresponding to different $C_{neg}$ values. Shadows in these figures indicate the 1$^{st}$ band gaps of the $A_0$
mode along the $x$ direction. From these results we can see that, when the $C_{neg}$ is not close to the boundaries of the unstable zone (e.g., $-15 \, nF$ and $-5 \, nF$), the EMM has quite good accuracy at frequencies below the 1st band gap. As the $C_{neg}$ approaches the unstable zone, the working frequency range of the EMM decreases. When the $C_{neg}$ is in the vicinity of the unstable zone (e.g., $-12.2 \, nF$ and $-10.4 \, nF$), the EMM only works at frequencies far below the band gap.

![Dispersion relations of the waves in structures composed of cells shown in Figure 2.5 corresponding to different $C_{neg}$ values. Dispersion curves of the flexural mode (i.e., the $A_0$ mode) obtained by using the EMM are also shown in these figures. Shadows indicate the 1st band gaps of the $A_0$ mode along the $x$ direction.](image)

2.5 Recent advances in wave control using piezoelectric materials

The tunable feature of the shunted piezoelectric patches has been widely exploited to control waves in the past, significant efforts have been achieved. In this section, some recent advances in this area are briefly introduced and discussed.

2.5.1 Adaptive metamaterials

O. Thorp et al. [106] first proposed to periodically arrange shunted piezoelectric patches on rods to form band gaps, in which energy transmission is forbidden. Later, this idea was followed by many researchers to control waves in beams [107, 108, 109, 3, 110, 111, 112] and plates [113, 114, 115, 116, 104]. Also, the concepts
Chapter 2. Tunable ability of shunted piezoelectrics

of adaptive/tunable metamaterials were proposed and became popular. Adaptive metamaterials commonly are composed of unit cells containing shunted piezoelectric transducers, an example is shown in Figure 2.8. The shunting impedance varies, including resistance, inductance and NC, as well as combination of these elements. The fundamental feature of these adaptive metamaterials is that their band gaps can be tuned by changing the shunting impedance value to cover desired frequency ranges [108, 109, 112, 3].

![Figure 2.8: (a) A structure with infinitely repeating unit cells, (b) a single unit cell containing shunted piezoelectric patches [110].](image)

2.5.2 Hybrid dispersive media

The hybrid medium contains a mechanical substrate, bonded piezoelectric patches and a network connecting all these patches, as shown in Figure 2.9. The idea to interconnect piezoelectric patches was firstly exploited by Dell’Isola et al. [117, 118, 119, 120]. This technique can be used to control vibration of structures [121, 122, 123]. Recently, this idea was adopted to control waves. These hybrid media are characterized by periodicity in both the electrical and mechanical domains, hence band gaps were observed in them [124]. Also at low frequency range, two different flexural modes exist in the hybrid plate, dispersion curves of these modes feature veering as the frequency increases from 0 Hz. This feature was exploited to improve the sound insulation performance of plates [125].

2.5.3 Reconfigurable structures

The reconfigurable structure has shunted piezoelectric transducers as part of its components. For example, P. Celli and S. Gonella [4] studied a modified hexagonal honeycomb with auxiliary cantilever elements, as shown in Figure 2.10. Each can-
2.5. Recent advances in wave control using piezoelectric materials

Figure 2.9: A hybrid media composed of a mechanical substrate and an piezoelectric elements interconnected through inductive elements [124].

A cantilever is instrumented with two thin piezoelectric elements shunted with NC. By differing the shunted NC values of piezoelectric patches the symmetry of the cell can be reconfigured to active different wave patterns of directivity.

Figure 2.10: Unit cell with its characteristic dimensions; the shaded regions correspond to piezoelectric patches. \( Z_{SH} \) is the equivalent electrical impedance of the shunting circuit [4].

2.5.4 Gradient index interfaces

During this thesis, the piezoelectric materials were also exploited to realize Gradient Index (GRIN) interfaces by other research groups [126, 127]. For example, Y. Chen et al. [126] designed a GRIN metamaterial-enhanced flexural wave sensing system to increase the quality and quantity of the flexural wave measurement data. The concept and realization of the GRIN interface is shown in Figure 2.11. The GRIN feature was realized through the piezoelectric patches shunted with NC. The shunting NC value along the beam smoothly approaches the unstable zone from the soften side. Amplitudes of flexural waves gains as they propagate in the beam.
Chapter 2. Tunable ability of shunted piezoelectrics

In this chapter, we studied the tunable feature of the piezoelectrics shunted with NC. We derived 5 independent parameters to describe the material properties of the shunted piezoelectric patches, which are transverse isotropic. These parameters include 2 effective Young’s moduli, 1 effective shear modulus and 2 effective Poisson’s ratio. Except the shear modulus, the other 4 parameters can be tuned by varying the shunting $C_{neg}$ within a large range. Due to this tuning feature, when the shunted piezoelectric patches are bonded with the passive plate, the dynamical properties of the whole cell also become tunable. We obtained the effective area density and bending stiffness of the cell by using an EMM. We demonstrated that the cell can be either softened or stiffened by tuning the $C_{neg}$ within the stable zone.

We also discussed the accuracy of the EMM. We obtained the dispersion curves of the flexural mode (i.e., the $A_0$ mode) in structures composed of identical cells shown in Figure 2.5 by using the EMM. Then, we compared these curves with those obtained by using the FEM and B.C. Results show that the EMM has quite good accuracy at frequencies below the 1st band gap of the $A_0$ mode when the shunting $C_{neg}$ is not close to the unstable zone. The working frequency range of the EMM decreases as the $C_{neg}$ approaches to the unstable zone. Consequently, the EMM is
only valid at frequencies far below the band gap in vicinity of the unstable zone.

The adaptive feature of the shunted piezoelectrics revealed in this chapter enables us to design metamaterials with tunable band gaps or with reconfigurable cell symmetry and even GRIN devices.
Chapter 3

Flexural waves focusing through shunted piezoelectric patches

Contents

3.1 Piezo-lens for wave focusing ........................................... 46
3.2 Numerical model and energy analysis ................................. 48
  3.2.1 Finite element model of piezo-mechanical systems .......... 48
  3.2.2 Energy analysis for harmonically excited thin plate ...... 49
3.3 Numerical results ....................................................... 49
  3.3.1 Focusing effect ..................................................... 49
  3.3.2 Adaptive ability ................................................... 52
  3.3.3 Performances of piezo-lens at different frequencies ...... 53
  3.3.4 Performances of piezo-lens for flexural waves excited by different types of sources ........................................ 58
  3.3.5 Double piezo-lenses configuration ............................. 64
3.4 Conclusion and discussion ............................................. 67

In this chapter, efforts are made to focus flexural waves in thin plates by using shunted piezoelectric patches. Piezoelectric patches are periodically bonded on the upper and lower surfaces of the host plate in a collocated fashion to form a flat GRadient INdex (GRIN) piezo-lens. Negative capacitance circuits are used to shunt these piezoelectric patches. An analytical relation between the negative capacitance value and the effective refractive index of the piezo-mechanical system is established according to the Effective Medium Model (EMM) developed in Section 2.2 of Chapter 2. The piezo-lens is designed based on this relation. The focusing effects are studied by using the Finite Element Method (FEM). In the numerical study, the focusing effect of piezo-lens is verified and the adaptive ability is demonstrated. The performances of piezo-lens at different frequencies are then analyzed. After that, the performances of piezo-lens for flexural waves generated by different types of sources at different frequencies are estimated. A double piezo-lenses configuration is proposed to focus waves excited by near field point forces. Discussions on the potential applications of the piezo-lens are also made in the conclusion part. This chapter is organized into four sections. Section 3.1 introduces the piezo-lens model and describes the designing process of the piezo-lens. Section 3.2 introduces the numerical model of the piezo-mechanical system and the energy analyses used in
analyzing the numerical results. Section 3.3 presents the numerical results. Section 3.4 summaries the remarkable conclusions and gives corresponding discussions.

3.1 Piezo-lens for wave focusing

The model and designing process of the piezo-lens are introduced in this section. The piezo-lens is obtained by periodically bonding piezoelectric patches on the surfaces of a host aluminum plate in a collocated fashion, as depicted in Figure 3.1(a). The host plate is lying in the $x-y$ plane and occupying the spatial region $-h_b/2 \leq z \leq h_b/2$. The piezo-lens zone could be divided into a 14-by-6 array of cell, the piezoelectric patches in each of these cells are shunted with NC circuits, as shown in Figure 3.2.

![Figure 3.1: (a) The harvesting system with piezo-lens and (b) the gradient variation profile of the refractive index $n(y)$.](image)

As proposed by [16], a flat GRIN lens to focus flexural waves in thin plate can be obtained if the refractive index for flexural wave inside the lens zone fulfills a hyperbolic secant function:
3.1. Piezo-lens for wave focusing

\[ n(y) = n_0 \cdot \text{sech}[\alpha(y - \beta)] \]  
\( \quad \text{ (3.1)} \)

in which, \( n_0 \) represents the refractive index of the background plate, \( \alpha \) is the gradient coefficient and \( \beta \) represents the \( y \) coordinate of the symmetry axis of the refractive index profile, as shown in Figure 3.1(b). Waves incident into the lens from the \( x \) direction will be focused at a focal point on the \( y = \beta \) line, with a focal length represented as \( f = \pi/2\alpha \).

In the considered piezo-lens, the above variation of the refractive index is approximately realized in a piecewise form by designing the shunting negative capacitance values of different cells. According to Equation (3.1), the refractive index only varies in the \( y \) direction. Thus, in the piezo-lens, the shunting negative capacitance values are equal in a same row (the \( x \) direction) but will be different in a same column (the \( y \) direction). To determine the required shunting negative capacitance value in each row, we used the EMM developed in Section 2.2.

According to the EMM, the properties of the cell are described by the effective area density \( \rho_{\text{eff}} \) and effective bending stiffness \( D_{\text{eff}} \). Therefore, the effective refractive index for flexural waves incident from the background plate into the cell can be expressed as:

\[ n_{\text{eff}} = \left( \frac{\rho_{\text{eff}} D_b}{\rho_b h_b D_{\text{eff}}} \right)^{1/4} \]  
\( \quad \text{ (3.2)} \)

in which, \( D_b = \frac{E_b h_b^3}{12(1-\nu_b^2)} \) is the bending stiffness of the host plate and \( E_b, \nu_b \) denote the Young’s modulus and the Poisson’s ratio of the host plate, respectively; \( \rho_b \) is the density of the host plate.

Figure 3.3 illustrates the variation of the effective refractive index with the applied negative capacitance value \( C_{\text{neg}} \). In the Figure, \( C_P^T \) is the intrinsic capacitance of the piezoelectric patch at constant stress. It is observed that the effective refractive index can be increased or decreased within a large range by varying the negative capacitance value in the stable zone.

![Figure 3.3: Variation of effective refractive index with negative capacitance value.](image-url)

Using the relationship in Equation (3.2), the piezo-lens is designed in three steps. In the first step, we choose the parameters \( \alpha \) and \( \beta \) in the refractive index profile.
in Equation (3.1) to design the location of the focal point. In the second step, the required refractive index for each row in the lens zone is obtained by substituting the central \( y \) coordinate of the row into Equation (3.1). In the last step, the required refractive index for each row is fulfilled by choosing the negative capacitance value according to Equation (3.2).

3.2 Numerical model and energy analysis

3.2.1 Finite element model of piezo-mechanical systems

To study performances of the piezo-lens, a Finite Element (FE) model of the piezoelectric system shown in Figure 3.1(a) is developed. In the FE model, the structures are discretized by 3D quadratic Lagrange elements. Each of the nodes corresponding to the piezoelectric patches has three mechanical Degrees Of Freedom (DOFs) and a voltage DOF. The equilibrium equations for the discretized fully coupled piezoelectric system are:

\[
M_{dd} \ddot{d} + K_{dd} d + K_{dV} V = F
\]

\[
-K_{dV}^T \dot{d} + K_{VV} V = Q
\]

Here, \( d \) and \( V \) represent the structural and voltage DOFs, respectively; \( F \) and \( Q \) are the mechanical forces and charges respectively.

The equations in (3.3) are rewritten under following considerations: (i) the voltage DOFs in the piezoelectric patches can be partitioned into DOFs inside the patches, DOFs on the free electrodes of the patches and DOFs on the bonding surfaces; (ii) the voltage DOFs on the bonding surfaces are grounded, thus the corresponding equations and columns are directly removed; (iii) there is no charge source inside the piezoelectric patches, thus the internal voltage DOFs can be eliminated by exact static condensation; (iv) as the DOFs on one electrode have identical voltages, the voltage DOFs on the free electrode of each piezoelectric patch are reduced such that only one master voltage DOF remains on the free electrode per patch. Consequently, the governing equations can be rewritten as below:

\[
\begin{bmatrix}
M_{dd} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{d} \\
\dot{V}_L
\end{bmatrix} +
\begin{bmatrix}
H_{dd} & H_{dL} \\
-H_{dL}^T & C_L
\end{bmatrix}
\begin{bmatrix}
d \\
V_L
\end{bmatrix} =
\begin{bmatrix}
F \\
Q_L
\end{bmatrix}
\]

In which, the matrix \( C_L \) is diagonal, each of its diagonal elements represents the blocked intrinsic capacitance of a piezoelectric patch in the piezo-lens; \( V_L \) is the master DOFs on the free electrodes of the patches in the lens; \( Q_L \) is the charges flowing to the patches in the lens. More details about above process could be found in Appendix B or [128].

The NC circuits are connected into the piezoelectric cells by considering following electrical boundary conditions:

\[
Q_L = -C_{neg} V_L
\]
here, $C_{neg}$ is a diagonal matrix of the designed NC values in Section 3.1.

### 3.2.2 Energy analysis for harmonically excited thin plate

To better understand the underlying physics and the focusing effects of the piezo-lens, energy analyses including power flow or kinetic energy have been conducted.

The very essential consequence of wave propagation is the power diffusion in the medium. The power diffusion pattern can be characterized by power flows. Accordingly power flows are useful in depicting the wave paths inside the lens zone. In frequency domain, the power flow in an elastic medium is defined as:

$$ I = \frac{1}{2} \omega \cdot re(-\sigma \cdot w^*) $$

(3.6)

here, $(\cdot)^*$ represents conjugate value and $re(\cdot)$ means only the real part is retained. Therefore, the total mechanical energy flux $I_{tot}$ of the cross-section of the host plate is:

$$ I_{tot} = \int_{-\frac{h_p}{2}}^{\frac{h_b}{2}} I \, dz $$

(3.7)

For harmonically excited structures, the time averaged kinetic energy is:

$$ W_k = \frac{1}{4} \omega^2 \rho w^{*T} \cdot w $$

(3.8)

It will be used in estimating the focusing effects.

### 3.3 Numerical results

#### 3.3.1 Focusing effect

The piezo-lens in the simulations has dimensions of 0.24 m $\times$ 0.58 m in the $x - y$ plane as depicted in Figure 3.1(a). The Perfectly matched layers are used surrounding the plate to avoid wave reflections at the boundaries [129, 130]. The geometry parameters of the unit cell are: $l_b = 0.04$ m, $l_p = 0.035$ m, $h_b = 0.005$ m and $h_p = 0.001$ m. Material parameters of the host plate and piezoelectric patches are listed in Appendix A. Light damping amount to a hysteresis coefficient of 0.1% was applied.

In the simulations to verify the focusing effect of piezo-lens, flexural waves were excited by a surface line harmonic transverse force located 0.1 m away from the left boundary of the lens. The parameters of piezo-lens were set as $\alpha = \pi/0.6$ and $\beta = 0$. With these settings, theoretically flexural waves will be focused at a distance $f = 0.3$ m on the $y = 0$ line.

Figure 3.4 shows the normalized power flows in the host plate without and with piezo-lens at 2000 Hz. In the Figure, the arrows indicate the directions and magnitude of the power flows. It can be observed that the magnitude of the power flows in the plate reduces when the waves are incident into the piezo-lens zone, and
the reduction becomes more obvious at locations far from the symmetry axis $y = 0$. There are two reasons for this phenomenon. On one hand, the refractive indexes inside the piezo-lens are designed to fit Equation (3.1). Thus, at the left and right interfaces between the piezo-lens and the background plate, the impedances only match on the symmetry axis, and the impedance discontinuities increase with the distance away from the symmetry axis. As a result, part of the incident waves will be reflected at the interfaces and the reflections gain with the distance away from the symmetry axis. On the other hand, in the piezo-lens zone, the piezoelectric patches are bonded on the surfaces of the host plate, part of the incident power in the host plate will flows into these patches. It can also be observed that, except near the upper and lower boundaries, power inside the piezo-lens zone is flowing toward the designed focal point. This verifies that the piezo-lens is bending the flexural waves as designed. It should be noted that the outgoing power near the upper and lower boundaries is not unique to the piezo-lens, it is a characteristic of the flat GRIN lens, more details can be found in the supplementary document.

![Figure 3.4: Normalized power flows in the host plate at 2000 Hz (a) without and (b) with piezo-lens. Black cross indicates the designed focal point.](image)
3.3. Numerical results

Figure 3.5 is the normalized kinetic energy distribution pattern after the piezo-lens at 2000 Hz. It can be observed that due to the wave bending effect, most of the incident energy is concentrated inside a limited zone around the designed focal point. To characterize this energy concentration effect, a energy concentration zone is defined. The kinetic energy inside this zone satisfies following condition:

\[ \frac{W_k}{\max(W_k)} \geq 0.8 \]  

(3.9)

The condition in Equation (3.9) indicates that inside the energy concentration zone, the kinetic energy is larger than or equal to 0.8 times the maximum kinetic energy after the lens. This zone is highlighted by solid line in this paper. For example, the energy concentration zone at 2000 Hz is illustrated in Figure 3.5.

Figure 3.5: Normalized kinetic energy distribution after the piezo-lens at 2000 Hz. Black cross indicates the designed focal point, solid line indicates the energy concentration zone.

Figure 3.6: Energy enhancement ratio distribution after the piezo-lens at 2000 Hz. Black cross indicates the designed focal point, dashed line indicates the energy enhancement zone.
Compared to the case without lens, the energy concentration effect after the piezo-lens could enhance the energy in particular zone. To illustrate this effect, an energy enhancement ratio \( \text{Eer} \) is defined as the ratio of the kinetic energy of plate with lens to the kinetic energy of plate without lens:

\[
\text{Eer}(x, y) = \frac{W_k^l(x, y)}{W_k^w(x, y)}
\]  

(3.10)

In Figure 3.6, the \( \text{Eer} \) after the piezo-lens at 2000 Hz is illustrated. According to Figure 3.5 and Figure 3.6, it can be observed that the energy is significantly enhanced inside the energy concentration zone. Similarly, to characterize this energy enhancement effect, an energy enhancement zone is defined. In this zone, the \( \text{Eer} \) satisfies following condition:

\[
\text{Eer}(x, y) \geq 2
\]  

(3.11)

The energy enhancement zone is depicted by dashed line in this paper, as can be observed in Figure 3.6.

### 3.3.2 Adaptive ability

In the piezo-lens, different refractive index profiles can be fulfilled by just tuning the shunting negative capacitance values, i.e., with the same geometry configuration, the piezo-lens can focus waves at different locations. This adaptive ability is demonstrated here, and in these simulations the flexural waves are excited by a surface line harmonic transverse force located 0.1 m away from the left boundary of the lens.

The piezo-lens can focus waves at different locations in the \( x \) direction. To verify this, the parameter \( \beta \) is fixed as \( \beta = 0 \), but the parameter \( \alpha \) is chosen as \( \alpha = \pi/0.6 \), \( \alpha = \pi/0.8 \) and \( \alpha = \pi \), to focus waves at distances 0.3 m, 0.4 m and 0.5 m on the \( y = 0 \) line, respectively. The focusing effect of these three piezo-lenses at 2000 Hz are illustrated in Figure 3.7. In the left panel are shown the normalized kinetic energy distribution after the piezo-lenses. In the right panel are depicted the normalized kinetic energy of the plates with piezo-lenses along the \( y = 0 \) line, the results of plates without lens are also illustrated as references. The adaptive ability of the piezo-lens in the \( x \) direction can be observed from the results in Figure 3.7. It can also be observed that the energy concentration zone enlarges as the focal length increases, indicating that the energy will be less concentrated at a larger distance.

The piezo-lens can also focus waves at different locations in the \( y \) direction. For example, fix the parameter \( \alpha \) as \( \alpha = \pi/0.6 \) and chose the parameter \( \beta \) as \( \beta = 0.12 \), \( \beta = 0 \) and \( \beta = -0.08 \) to focus waves at a distance 0.3 m on the \( y = 0.12 \) line, \( y = 0 \) line and \( y = -0.08 \) line, respectively. Figure 3.8 shows the designed refractive index profiles and the focusing effects of the corresponding piezo-lenses at 2000 Hz. It can be observed that the piezo-lens is adaptive in the \( y \) direction. It should be noted that a larger distance between the symmetry axis of the refractive index profile (\( y = \beta \) line) and the central axis of the lens (\( y = 0 \) line) will results in less energy
concentration intensity and smaller energy enhancement zone, as illustrated in the figure.

3.3.3 Performances of piezo-lens at different frequencies

The focusing effects of piezo-lens at different frequencies are studied in this section. In the numerical simulations, the same excitation source and parameters of piezo-lens used in the simulations in Section 3.3.1 were adopted.

Figure 3.9 shows the focusing effects of piezo-lens at different frequencies. As expected the effectiveness of piezo-lens is limited within a certain frequency band. The lower limit frequency is dominated by the characteristic length of piezo-lens. At frequencies smaller than the lower limit, the corresponding wavelengths will be larger than the characteristic length of piezo-lens. At these circumstances, waves will bypass the piezo-lens by diffraction effect [7] and the piezo-lens will have poor focusing performance. The lower limit for the piezo-lens in this paper is around 100 Hz. At this frequency, the flexural wavelength is just a little longer than the piezo-lens’ length in the $y$ direction. As a result of diffractive effect, most of the waves will bypass the piezo-lens without being focused, leading to a quite large energy concentration zone and no energy enhancement zone. On the other hand, the upper limit frequency is dominated by the length of cell. At frequencies near the upper limit, the flexural wavelengths will be approximately equal to twice of the cell’s length, most of the incident energy will be reflected by the piezo-lens [100], results in poor performance. The upper limit in this paper is about 8000 Hz. At 8000 Hz, the wavelength is almost equal to twice of the cell’s length. At this frequency, even though an energy concentration zone can still be observed, but there is no energy enhancement zone due to the large reflection.

Inside the effective frequency band, the performances of piezo-lens are dependent on the frequencies. At frequencies from 1000 Hz to 6000 Hz, as the frequency increases, energy will be less concentrated and the energy concentration zone will shifts to the right-hand side of the designed focal point. Consequent energy enhancement zone will become longer in the $x$ direction. However, below 2000 Hz, energy is concentrated almost around the designed focal point. The wavelength at 2000 Hz is nearly four times of the cell’s length. Hence, the piezo-lens could focus energy near the designed point if the wavelength is larger than four times of the cell’s length.

There are several reasons that probably contribute to the shifting of focal location and less concentration of energy at higher frequencies inside the effective frequency band. Firstly, the wavelength is more comparable to the cell’s length at higher frequencies. In the piezo-lens, the variation of refractive index is realized in a piecewise form. At smaller wavelengths, this variation will be less smooth. Secondly, the homogenized model of piezoelectric cell used in the designing process becomes less accurate at higher frequencies [104]. Accordingly, the actual refractive index variation profile in the piezo-lens will deviates from the designed one at these frequencies. Since the focal location of waves is dependent on the refractive index
Figure 3.7: Left panel: normalized kinetic energy distributions after the piezo-lens for different focal locations at 2000 Hz, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. Right panel: normalized kinetic energy along $y = 0$ lines for different focal locations at 2000 Hz, black crosses indicate the designed focal points.
3.3. Numerical results

Figure 3.8: Left panel: designed refractive index profiles. Right panel: normalized kinetic energy distributions after the piezo-lens for different refractive index profiles at 2000 Hz, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.

(a) $\alpha = \pi/0.6, \beta = 0.12$

(b) $\alpha = \pi/0.6, \beta = 0$

(c) $\alpha = \pi/0.6, \beta = -0.08$
Figure 3.9: Normalized kinetic energy distributions after the piezo-lens at different frequencies. Black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.
3.3. Numerical results

The performances of piezo-lens at different frequencies can also be predicted by the maximum energy enhancement ratio \( \max(Eer) \) after the lens. The \( \max(Eer) \) after an ideal lens and a piezo-lens at different frequencies are illustrated in Figure 3.10. The ideal lens is used as a reference, it has the same dimensions with the piezo-lens and is designed by consecutively varying the Young’s modulus in the lens zone. In the figure, a larger \( \max(Eer) \) generally indicates better focusing effect. This can be seen from the fact that at lower limit frequency (100 Hz) and at upper limit frequency (8000 Hz), the \( \max(Eer) \) after the piezo-lens are much smaller than most of those at frequencies inside the effective frequency band. As the frequency approaches to the upper limit, the performance of piezo-lens declines, this is coincident with the results revealed in Figure 3.9. Especially near the upper limit at 7000 Hz, the performance of piezo-lens is already poor. Therefore utilization near the upper limit should be avoided for better applications. The piezo-lens is commonly less effect than the ideal lens inside the effective frequency band. This is reasonable since the piezo-lens has discrete configuration, it will reflects more energy than the ideal one.

According to the results above, the effective frequency band of piezo-lens can be extended by increasing the length of piezo-lens and/or decreasing the length of cell. To demonstrate this, a new piezo-lens model with smaller cells was designed. The new piezo-lens is composed of 19-by-8 array of piezoelectric cells, each unit piezoelectric cell has the dimensions illustrated in Table 3.1. The new piezo-lens has smaller cells but has more, therefore, there are little differences between the global dimensions of the new piezo-lens (0.24 m * 0.57 m) and the existing one.

![Figure 3.10: The maximum energy enhancement ratios at different frequencies](image)

Profile inside the piezo-lens, this deviation will make the flexural waves be focused away from the designed point. Thirdly, the anisotropy of piezoelectric cell could be more obvious at higher frequencies. The anisotropy of cell will cause the aberration of focus [131].

Figure 3.10: The maximum energy enhancement ratios at different frequencies.
Figure 3.11: The maximum energy enhancement ratios at different frequencies for the new piezo-lens with smaller cells.

Table 3.1: Geometry parameters of one unit piezoelectric cell in the new piezo-lens

<table>
<thead>
<tr>
<th>$l_b$</th>
<th>$h_b$</th>
<th>$l_p$</th>
<th>$h_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03 m</td>
<td>0.005 m</td>
<td>0.025 m</td>
<td>0.001 m</td>
</tr>
</tbody>
</table>

(0.24 m * 0.56 m). The parameters of the new piezo-lens are set as $\alpha = \pi / 0.6$ and $\beta = 0$, which are coincident with the settings in other simulations in this section. Figure 3.11 shows the maximum energy enhancement ratio after the new piezo-lens at different frequencies. Compared with the results in Figure 3.10, it is clear that the effective frequency band has been significantly broadened, from [1000, 6000] Hz to [1000, 11500] Hz.

### 3.3.4 Performances of piezo-lens for flexural waves excited by different types of sources

Flat GRIN lenses are originally designed to focus waves incident from the normal direction. Actually, any wave can be decomposed to have a component in the normal direction. Thus, a flat GRIN lens should have focusing effects for other kinds of incident waves to some extent. In this section, the performances of piezo-lens for obliquely incident plane waves and waves excited by point source are studied. The parameters of piezo-lens are set as $\alpha = \pi / 0.6$ and $\beta = 0$ in this section.
3.3. Numerical results

3.3.4.1 Oblique plane waves

In these cases, oblique plane waves are generated by a surface line harmonic transverse force with an angle $\theta$ against the left boundary of the piezo-lens. A positive $\theta$ represents waves are incident from the bottom left direction and a negative one represents waves are incident from the upper left direction, due to the symmetry only the positive $\theta$ will be considered.

Focusing effects at these cases are demonstrated by an example in Figure 3.12. For the oblique plane wave, energy is focused above the designed focal point. The deviation of the energy concentration zone from the designed location is expected. The wave incident with an angle can be decomposed into an $x$ component and a $y$ component. The $x$ component will be focused to a point and the $y$ component will deviates this real focal point away from the designed one in the $y$ direction. Similar with the normal incident wave case, as the frequency increases the energy concentration zone will shifts to the right and enlarges. Note that at higher frequency (6000 Hz), an extra focalization can be observed near the right piezo-lens boundary. The reason of this phenomenon is interpreted in Appendix C.

![Normalized kinetic energy distributions after the piezo-lens for waves incident with $\theta = 20^\circ$. Black cross represents the designed focal point, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.](image)

The influences of incident angle on the focusing effects at different frequencies are predicted in Figure 3.13. Inside the effective frequency band (from 1000 Hz to 6000 Hz), the overall trend is that the focusing effects decline with the increase of incident angle. Particularly at $\theta = 40^\circ$ case, the $\text{max}(Ecr)$ at frequencies inside the
Chapter 3. Flexural waves focusing through shunted piezoelectric patches

The effective frequency band are comparable with that at upper limit frequency (8000 Hz). The $\max(\text{Eer})$ at frequencies inside the effective frequency band should be much larger than that at limit frequencies if the piezo-lens has good performance. From this point of view, the piezo-lens already has poor effect at $\theta = 40^\circ$ case. Therefore, the incident angle of plane waves should be limited in a range, in our case here the range is about $[-40^\circ, 40^\circ]$.

The more specific influences of the incident angle on the focusing effects at 2000 Hz are illustrated in Figure 3.14. As the incident angle increases, the energy concentration zone will shifts upward and becomes smaller. Consequently, the energy enhancement zone also shifts upward and shrinks. According to Figure 3.13 and Figure 3.14, at the case with larger incident angle, even thought the energy is more concentrated, but the $\max(\text{Eer})$ at that case is more smaller. This is mainly caused by the reason that more energy will be reflected when the waves are incident with a larger angle.

3.3.4.2 Waves excited by point forces

In these simulations, flexural waves are generated by a point force located on the central axis of the piezo-lens ($y = 0$ line) with a distance $d$ from the left boundary of the piezo-lens. It is easy to understand that if the point force is enough far away from the piezo-lens, the waves incident upon the piezo-lens can be treated as plane waves, the piezo-lens should have focusing effects at these cases. For $d = 1.2$ m case, the distance is more than five times of the longest wave length of the considered
3.3. Numerical results

![Diagram](image)

(a) energy concentration zone

(b) energy enhancement zone

Figure 3.14: Influence of the incident angle on (a) the energy concentration zone and (b) the energy enhancement zone at 2000 Hz. Black cross represents the designed focal point.

frequency, it is assumed that this distance is enough far. The focusing effects at this case are illustrated in Figure 3.15. It can be seen that energy is effectively focused. Comparing the corresponding results in Figure 3.9 and Figure 3.15, the right shifting of the energy concentration zone is more obvious in Figure 3.15, indicating that the piezo-lens is more sensitive to frequency in point force case.

The influences of point force distance on the focusing effects at different frequencies are predicted in Figure 3.16. Inside the effective frequency band (1000 Hz to 6000 Hz), from an overall point of view, the focusing effects improve with the increase of distance. At distances larger than or equal to 0.3 m, the $\max(Eer)$ at frequencies inside the effective band are obviously larger than that at limit frequencies (100 Hz and 8000 Hz). Therefore the distance of the point force should be larger than 0.3 m for achieving acceptable focusing effects, which is almost one and half times of the longest wavelength of the considered effective frequencies.
Figure 3.15: Normalized kinetic energy distributions after the piezo-lens for waves excited by a point force located 1.2 m away from the left lens boundary, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.

Figure 3.16: Influence of the point force distance on the maximum energy enhancement ratio at different frequencies.
3.3. Numerical results

The variations of energy concentration zone and energy enhancement zone with the point force distance at 2000 Hz are illustrated in Figure 3.17. It can be observed that as the distance increases the energy will be focused more close to the designed focal point and be more concentrated. Consequently, the energy enhancement zone shifts to the left and shrinks.

![Figure 3.17: Influence of the point force distance on (a) the energy concentration zone and (b) the energy enhancement zone at 2000 Hz, black cross represents the designed focal point.](image)

The piezo-lens can also focus waves generated by point force away from the central axis. According to the performances of piezo-lens for oblique plane waves and waves excited by point forces on the central axis, it has limitations for the location of the paraxial point force. The vertical distance of the point force to the
left piezo-lens boundary should be at least one and half times larger than the longest wavelength and the vertical distance of the point force to the central axis should guarantee that the incident angles of the waves are smaller than 40°. Even with these limitations, the point force is available in a large zone. As an example, in Figure 3.18 are shown the focusing effects for waves excited by a point force located 1 m away from the piezo-lens and 0.2 m away from the central axis. As expected, the piezo-lens is effective in a large frequency band. Energy is concentrated away from the designed focal point and energy concentration zone shifts to the right with the increase of frequency. Similar to the oblique plane wave cases, multiple focalization is observed at higher frequency.

### 3.3.5 Double piezo-lenses configuration

The piezo-lens has good focusing effects for waves generated by point force located sufficient far away. Whereas, the wave fields generated by point force are scattering, energy reduces with the wave traveling distance. Even though energy can be significantly enhanced after the lens, magnitude of the energy could be still unacceptable. Thus, it is more practical to focus waves near the point source. The wave propagation in the flat GRIN lens zone is reversible, i.e., waves generated by point force located at the designed focal point will be adjusted to parallel with the central axis by the lens. According to this, a double piezo-lenses configuration is proposed.
3.3. Numerical results

Figure 3.19: Normalized kinetic energy distributions after the double piezo-lenses configuration for near field point force at different frequencies, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.

In this configuration, an identical piezo-lens depicted in Figure 3.1(a) is positioned just after the original one. The first piezo-lens is used to bend the waves generated by a point force at the designed focal point to be parallel with the central axis, the second piezo-lens is then used to focus these waves. The efficiency of this proposition is demonstrated by an example below. Set the parameters as $\alpha = \pi/0.6$ and $\beta = 0$ for each piezo-lens in the double-lenses configuration. The focal point on the left side is 0.06 m away from the left boundary on the $y = 0$ line and a point force is located at it. The focusing effects at different frequencies are demonstrated in Figure 3.19.

Due to the adaptive ability, the double piezo-lenses configuration can focus energy at different locations as demonstrated in Figure 3.20. In these examples, the two piezo-lenses in a double-lenses configuration are identical. Parameter $\alpha$ is set as $\alpha = \pi/0.6$ for all the double piezo-lenses configurations, but parameter $\beta$ is chosen differently as $\beta = 0.12$, $\beta = 0$ and $\beta = -0.08$ to focus waves at a distance 0.3 m on the $y = 0.12$ line, $y = 0$ line and $y = -0.08$ line, respectively. Flexural waves are excited by point forces located at the corresponding left focal points at these three cases as depicted in the figure.
Figure 3.20: Normalized kinetic energy distributions after the double piezo-lenses configurations at 2000 Hz for different focal locations, black crosses indicate the designed focal points, red points indicate the locations of the point forces, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively.
3.4 Conclusion and discussion

The most remarkable conclusions obtained from the numerical results in this chapter are summarized below with corresponding discussions.

- The piezo-lens is effective in a frequency band, the lower limit frequency is dominated by the characteristic length of the piezo-lens, and the upper one is limited by the length of cell. The effective frequency band can be extended by decreasing the length of cell. Inside the effective frequency band, the focal zone will shift toward right and enlarges as the frequency increases. But energy can almost be focused near the designed location if the wavelength is larger than four times of the cell’s length.

Large effective frequency band makes the piezo-lens quite promising in broadband energy harvesting systems [132].

- With the same geometry configuration, the piezo-lens can focus waves at different locations by tuning the shunting negative capacitance values.

This adaptive ability distinguishes the piezo-lens from other existing GRIN lenses. With this ability, the piezo-lens can be used in SHM to simultaneously monitor a large region [133].

- The piezo-lens is effective for obliquely incident plane waves and waves excited by point forces in a broadband. The incident angle is available in a broad range and the point force can be located in a large zone.

In the existing literatures, flat GRIN lenses were only verified to be effective in focusing plane waves incident from the normal direction. The effectiveness of piezo-lens for waves generated by different types of sources demonstrates that the piezo-lens can be applied in other cases.

- The proposed double piezo-lenses configuration is effective for waves excited by point forces located in the near field in a large frequency band and can concentrate energy at different locations by adjusting the negative capacitance values.

The double piezo-lenses configuration is more practical for point source cases. The characteristics summarized above make it useful in broadband energy harvesting system and SHM.
Chapter 4

Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Contents

4.1 Introduction .................................................. 69

Part A. Concept of exploiting piezo-lens in wave energy harvesting 71

4.2 Full finite element model .................................... 73

4.3 Wave energy harvesting performances of the system incorporating the piezo-lens ........................................ 74
  4.3.1 Enhancement of the harvested power ......................... 74
  4.3.2 Adaptive feature ............................................. 77

Part B. Harvesting energy from transient elastic waves using the piezo-lens and SSHI-based techniques .................. 79

4.4 Harvesters .................................................... 79

4.5 Reduced finite element model ................................ 82

4.6 Numerical results .............................................. 85
  4.6.1 Comparison between the standard DC and SSHI-based devices .................. 85
  4.6.2 Effects of piezo-lens on transient waves and harvesting ........... 87
  4.6.3 Energy balance ............................................. 90
  4.6.4 Practical application considerations .......................... 92

4.7 Conclusions of Part A and B .................................. 94

4.1 Introduction

Small electronic devices are increasingly employed in many applications in the aerospace, transport and civil engineering fields. Such devices typically require continuous low power supply [134]. This fact motivates massive researches dedicated to the transformation of ambient energy into electricity. Due to the ubiquitous presence of vibration in structures, extensive efforts have been made to harvest vibration energy through piezoelectric, electromagnetic and electrostatic transducers [135]. In practical applications, vibration levels can be low [136] and the vibration
energy is typically distributed over a broad frequency band, consequently a lot of these researches were dedicated to obtain higher harvesting efficiency and to extend the operating frequency band of the harvesting system. Examples include (i) tuning the harvesting system through passive or active methods to match the operating frequency with the environment [136, 137]; (ii) exploiting nonlinear mechanical mechanisms to widen the operating frequency band [138] or nonlinear circuits to improve the extracted power from the transducers [139, 140, 141, 142, 143, 144, 145, 146, 147]; (iii) using phononic crystals [148], metamaterials [3] or acoustic black holes [53] to increase energy densities near the harvesters.

Apart from the mechanical approaches, nonlinear energy extraction circuits have been proposed to boost the harvested energy when piezoelectric transducers are used. The introduced nonlinear part is termed Synchronized Switch Harvesting on Inductor (SSHI) interface, which is composed of a switch and an inductor in series. The switch is turned off when the voltage of the piezoelectric transducer reaches a minimum or a maximum. Under this situation, the inductance and the capacitance constitute an oscillating circuit with a period much smaller than the mechanical one. After a very short time equals to a half of the period of the oscillating circuit, the switch is turned on again. Consequently, the voltage is inverted. The SSHI interface only requires a small amount of power to control the switch, it could be self-powered [146].

The SSHI interface is firstly proposed and studied by D. Guyomar et al. [139], and has drawn considerable attention these years. It is connected with the piezoelectric element in parallel to form a parallel SSHI harvesting system [143, 139, 140, 141], or in series to obtain a series SSHI harvesting system [143, 140]. Researches show that the SSHI interface can improve the converted energy by the piezoelectric transducer from mechanical to electrical form and reduce the backward conversion [145]. In a weekly coupled electromechanical system, the harvested power is boosted by 4 to 9 times when compared to a standard DC technique proposed by G. Ottman et al. [149]. In a strongly coupled case, these techniques will produce almost equivalent power but the required piezoelectric materials for the SSHI-based techniques are much less [139]. Based on the SSHI interface, several improved techniques have been proposed, each of them addresses a particular concern. Such as the Double Synchronized Switch Harvesting (DSSH) [142] and the Enhanced Synchronized Switch Harvesting (ESSH) [144] are proposed to obtain load independent techniques. The Adaptive Synchronized Switch Harvesting (ASSH) [147] technique is developed to harvest energy in multi-modal vibration situations. The characteristics, advantages and drawbacks of these SSHI-based techniques are discussed in details by D. Guyomar et al. [146].

Harvesting vibration energy in structures is well studied, however limited effort has been devoted to harvest energy from traveling waves. Traveling waves are common in built-up structures, since the power in these structures transmits from one component to another especially at higher frequency bands [150, 151]. In addition, waves will propagate away from the source when structures are under non stationary excitations, they are attenuated by damping and/or radiation thus are not reflected
back to the source to form standing waves. It is important to develop harvesting systems to obtain energy from waves in those cases.

To harvest energy from traveling waves, one of the main challenges is that the amount of harvested energy could be very low. In recent years, to increase harvested energy from traveling waves, several innovative harvesting systems have been developed [152, 153, 154, 8]. The fundamental idea is to steer waves to increase the energy densities at particular positions and harvest there. For examples, an elliptical acoustic mirror or a parabolic acoustic mirror is used to focus waves in the systems proposed by [152, 153, 154]; in [154], an artificial periodic array with a defect is designed to localize energy at specific frequencies, an acoustic funnel formed by arrays of acoustic scatters is developed to guide waves into a narrow channel. However, the harvesting circuits in these studies are simply represented by resistive ones in order to focus analysis in the mechanical part of the system. This is acceptable from an academic point of view but not adequate in practice since most of the real-life low power electronic devices require DC, the actual harvesting circuit always contains a AC-to-DC converter and maybe some other nonlinear parts, which can’t be simplified as a linear resistive load. Also the above introduced systems are all passive, they have no ability to adapt themselves to the environment changes.

Different from the aforementioned methods to steer waves, an adaptive piezo-lens is proposed to focus waves in Chapter 3. The piezo-lens is composed of a host plate and several surface-bonded piezoelectric patches. These patches are shunted with negative capacitance (NC) circuits. The spatial variation of refractive index in the piezo-lens zone is designed to fulfill a hyperbolic secant profile. Results show that the piezo-lens can focus flexural waves near a designed point in a broad frequency band. Thus the piezo-lens has large potential to be exploited in developing advanced harvesting systems for waves.

In this Chapter, the piezo-lens is used to improve energy harvesting from traveling waves. The whole chapter is divided into two parts. Part A discusses the concept and effect of using piezo-lens in wave energy harvesting. In this part, Section 4.2 introduces the full finite element model of the studied piezoelectric system. Section 4.3 demonstrates the effects of piezo-lens on harvesting. The second part (Part B) studies the harvesting performances of systems composed of the piezo-lens and SSHI-based harvesters. The analyzed harvesters are introduced in Section 4.4. Corrected reduced finite element models are developed to predict the transient responses of the systems in Section 4.5. The performances of the harvesting systems are studied and discussed in Section 4.6. Finally, conclusions of Part A and B are given in Section 4.7.

Part A. Concept of exploiting piezo-lens in wave energy harvesting

It is demonstrated in Chapter 3 that the piezo-lens can concentrate energy inside a limited zone around the designed focal point and the location of this energy concen-
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

The concentration zone can be adjusted by tuning the NC values. These two qualities can be of great practical interest for wave energy harvesting. In particular, the pre-designed location of the zone with a high level of energy density identifies the optimal locations for harvesting. Besides, even though the piezo-lens is initially designed to focus waves from the Ox direction, and when the incident direction is changed, waves will be focused away from the designed focal point [155], where the transducer for harvesting is placed; but by exploiting the tunable feature of the piezo-lens, the location of the energy concentration zone may be adjusted back to the desired site.

![Diagram of the harvesting system incorporating the adaptive piezo-lens](image)

Figure 4.1: (a) The harvesting system incorporating the adaptive piezo-lens and (b) the elementary unit in the piezo-lens.

To explore the concept of using piezo-lens in wave energy harvesting, a system incorporating a piezo-lens and a harvester is proposed, as illustrated in Figure 4.1(a). The harvester is composed of a piezoelectric patch bonded on the upper surface of the plate at the designed focal point and a connected resistor, as illustrated in Figure 4.2, in which, the piezoelectric patch is equivalently represented by a current source $I_{eq}(t)$ and a capacitance $C_h$ with the value equal to the blocked intrinsic capacitance (i.e. the capacitance when the patch is at constant strain) [145]. Since the output $V_h(t)$ of the harvester in Figure 4.2 is an AC voltage but most of the real-life low-
power electronic devices require DC input, this AC device has limited practical value. However, the amplitude of the dissipated power by the resistor could be used as a metric to estimate the potential power that could be converted and harvested via the piezoelectric transducer. Thus, the AC device is useful in evaluating the performances of harvesting systems especially with complicated mechanical parts and is used in many studies [148, 53, 152, 153, 154, 156]. Under harmonic excitations, the amplitudes of the output voltage and harvested power (namely the dissipated power by the resistor) are:

\[ \tilde{V}_h = -j\omega \tilde{Q}_h R, \quad P = \frac{\tilde{V}_h^2}{R} \]  \hspace{1cm} (4.1)

\( \tilde{Q}_h \) is the amplitude of the charge flowing to the piezoelectric patch.

\[ I(t) = -\dot{\tilde{Q}}_h(t) \]

\[ I_{eq}(t) \]

\[ C_h \]

\[ V_h(t) \]

\[ R \]

\[ \text{PZT} \]

Figure 4.2: An AC device

4.2 Full finite element model

Following the processes introduced in Appendix B, we developed the finite element model for the harvesting system shown in Figure 4.1(a):

\[
\begin{bmatrix}
M_{dd} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{d}_L \\
\ddot{d}_h \\
\ddot{d}_L
\end{bmatrix}
+ 
\begin{bmatrix}
H_{dd} & H_{dL} & H_{dh} \\
\frac{-H_{dL}^T}{C_L} & 0 \\
\frac{-H_{dh}}{C_h} & 0
\end{bmatrix}
\begin{bmatrix}
d_L \\
V_h \\
V_h
\end{bmatrix}
= 
\begin{bmatrix}
F \\
Q_L \\
Q_h
\end{bmatrix}
\]  \hspace{1cm} (4.2)

in which, matrices or vectors with subscript \( L \) are related to piezoelectric patches in the piezo-lens, those with subscript \( h \) are linked to the patch for harvesting. The matrices \( C_L \) and \( C_h \) are diagonal, each of their diagonal elements represents the blocked intrinsic capacitance of a piezoelectric patch; \( V_L, V_h \) are the master DOFs on the free electrodes of the patches; \( Q_L, Q_h \) are the charges flowing to the patches.

Under harmonic excitations, Equations (4.2) are simplified as:

\[
\begin{bmatrix}
-\omega^2 M_{dd} + H_{dd} & H_{dL} & H_{dh} \\
\frac{-H_{dL}}{C_L} & 0 \\
\frac{-H_{dh}}{C_h} & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{d}_L \\
\ddot{d}_L \\
\ddot{d}_L
\end{bmatrix}
= 
\begin{bmatrix}
\ddot{F} \\
\ddot{Q}_L \\
\ddot{Q}_h
\end{bmatrix}
\]  \hspace{1cm} (4.3)
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

the symbol \( \tilde{\cdot} \) denotes the complex amplitude in the frequency domain.

The AC device in Figure 4.2 is introduced into the harvesting system described by Equation (4.3) through the first relation in Equation (4.1). The NC circuits in the piezo-lens are introduced through the following electrical boundary conditions:

\[
\tilde{Q}_L = -C_{neg}\tilde{V}_L
\]

(4.4)

here, \( C_{neg} \) is a diagonal matrix of the designed NC values for the cells composing the piezo-lens.

4.3 Wave energy harvesting performances of the system incorporating the piezo-lens

The focusing performances of the piezo-lens demonstrated in Chapter 3 lead to two major expectations: (i) the designed focal point of the piezo-lens may be a good choice to place the harvester to obtain boosted power in a wide frequency band; (ii) the tunable feature of the piezo-lens could be exploited to cope with the change of wave incident direction. These expectations are numerically verified in this section.

In the numerical studies, the dimensions of the piezo-lens are \( 0.24 \text{ m} \times 0.58 \text{ m} \), as depicted in Figure 4.1(a). The Perfectly matched layers are used surrounding the plate to avoid wave reflections at the boundaries \([129, 130]\). The geometry parameters of the piezoelectric cells and piezoelectric patch for harvesting are given in Table 4.1, meanings of the symbols used in table 4.1 are given in Figure 4.1(b). Material parameters of the host plate and piezoelectric patches are listed in Appendix A. Line harmonic transverse forces are used to generate plane waves. Note that, in the simulations, without further information, the waves are incident into the piezo-lens from the \( Ox \) direction.

<table>
<thead>
<tr>
<th></th>
<th>( l_b )</th>
<th>( h_b )</th>
<th>( l_p )</th>
<th>( h_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>piezoelectric cell</td>
<td>0.04 m</td>
<td>0.005 m</td>
<td>0.035 m</td>
<td>0.001 m</td>
</tr>
<tr>
<td>piezoelectric patch for harvesting</td>
<td>( \backslash )</td>
<td>( \backslash )</td>
<td>0.05 m</td>
<td>0.001 m</td>
</tr>
</tbody>
</table>

4.3.1 Enhancement of the harvested power

The piezo-lens’ parameters are chosen as \( \alpha = \pi/0.6, \beta = 0 \). These values are chosen because the energy will be more better concentrated if the focal point of the piezo-lens is designed on the central axis and is close to the lens boundary. To demonstrate the first expectation, the harvested power at different frequencies for different load resistances is shown in Figure 4.3 when the harvester is mounted at the focal point.
4.3. Wave energy harvesting performances of the system incorporating the piezo-lens

In the figure, the maximum harvested power in the case without lens is normalized to unity; only the results at the frequencies from 1000 Hz to 7000 Hz are shown, because the piezo-lens has better performance within this frequency band, as will be seen hereinafter. From Figure 4.3, it can be seen that the harvested power is substantially boosted over a wide frequency range when the piezo-lens is applied. To more precisely estimate how many times the power could be improved, a gain ratio is defined as the ratio of the harvested power with piezo-lens to that without lens. Figure 4.4(a) shows the gain ratios at different cases. One can point out that the gain ratio is nearly independent of the resistance. Due to this independence, the gain ratios at different frequency could be studied by setting the resistance as a constant value, as depicted in Figure 4.4(b). It can be observed that from 1000 Hz to 7000 Hz, the harvested power is more than doubled; particularly at some frequencies, the power is improved over 400% of the one corresponding to the case without lens; below 1000 Hz and above 7000 Hz, the harvested power is barely enhanced or even decreased, this is because the piezo-lens has poor focusing performances at the frequencies close to the lower and upper limits of the effective frequency band.

![Figure 4.3: Harvested power versus resistance and frequency.](image)

The results above show that harvesting at the focal point can obtain boosted power in a wide frequency band. However as the frequency increases, the energy concentration zone will become longer in the $x$ direction and shift to the right hand side along the central axis of the lens, as shown in Figure 4.5, in which, the cross indicate the designed focal point, zones inside the circles with solid lines are energy concentration zones defined as:

$$\frac{E_k(x, y)}{\max(E_k(x, y))} \geq 0.8$$

(4.5)

in which, $E_k(x, y)$ is the kinetic energy density after the lens. Results in Figure 4.5 imply that harvesting at different sites may lead to diverse harvesting performances, i.e. the focal point may not be the most suitable place to put the harvester. Accord-
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Figure 4.4: (a) Gain ratio of harvested power versus frequency and resistance; (b) gain ratio of harvested power versus frequency when $R = 1000 \, \Omega$

...it is important to study the harvesting performances at different locations.

The waves on both sides of the central axis of the piezo-lens are bended toward the axis, due to the symmetry of the lens, these waves meet each other on the central axis with the same phase, thus in the energy concentration zone, locations on the central axis have the highest energy density. Besides, if the harvester is located too far from the focal point (namely is outside all the energy concentration zones), the harvesting performance apparently will be poor. Based on these understandings, this part is focused on studying the axial location effect of the harvester. In the studies, three different locations are chosen to put the harvester, as illustrated in Figure 4.6. In case 1, the harvester is at the focal point; in case 3, the harvester is outside most of the energy concentration zones but still within the zones corresponding to frequencies larger than 5000 Hz; the harvester in case 2 is placed between the aforementioned two locations. The harvested power and gain ratios of these three cases are compared in Figure 4.7, the harvested power of case 1 at each frequency is normalized to unity. As expected, from 1000 Hz to 5000 Hz, case 1 has better...
4.3. Wave energy harvesting performances of the system incorporating the piezo-lens

Figure 4.5: Energy concentration zones at different frequencies. The cross indicates the designed focal point.

performance than case 2 and 3; on contrary, at frequencies above 5000 Hz, case 2 and 3 have better performances. However except at 7000 Hz, the relative differences between the harvested power of these three cases are less than 30%, they are small compared with the gain ratios, which are typically larger than 2 as can be seen in Figure 4.7(b). That is to say, harvesting at any of these three locations can yield considerably boosted power in a large frequency band, in other words, even though the energy concentration zone is shifting with the increasing of frequency, the harvesting is insensitive to the location as long as the harvester is placed on the central axis and is not too far from the focal point. Based on this conclusion and recalling that the focal point is a pre-designed location, it is reasonable to say that the designed focal point is the most suitable place for harvesting.

4.3.2 Adaptive feature

Except the energy concentration effect, another remarkable merit of the piezo-lens is its tunable feature. The spatial variation of the refractive index inside the piezo-lens zone is realized by designing the NC values of the cells at different locations. As the theoretical focal point of the lens is determined by the refractive index profile in the lens, thus by tuning the NC values, different variations of the refractive index could be realized to focus waves at different locations. This adaptability is verified in Section 3.3.2: by tuning the NC values, the energy can be concentrated at different locations along the $x$ or $y$ directions.

The adaptability of the piezo-lens can be exploited to deal with the environment changes. When the piezo-lens is used in the harvesting system depicted in Figure 4.1(a), it is primarily designed to focus waves from the $Ox$ direction, and the harvester is located at the focal point. However, in practice the directions of the incident waves may be changed - the waves maybe no longer incident from the $Ox$ direction, but from an oblique direction. Due to this modification, the waves are focused away from the originally designed location, as illustrated in Figure 4.8(a). Since the harvester is initially located at the designed focal point, and is not inside the current energy concentration zones, the deviation of the energy concentration...
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Figure 4.6: Harvesting at different positions, the point indicates the designed focal point.

zone will result in significant reduction of the harvested power, as can be observed in Figure 4.9. The tunable feature of the piezo-lens could be used to handle this kind of circumstances. When the waves are incident from an oblique direction, the present focal point corresponding to the oblique waves has the same \( x \) coordinate but different \( y \) coordinate with the designed one, as illustrated in Figure 4.8(a). Recalling that the \( y \) location of the focal point is connected with the parameter \( \beta \). Thus in order to adjust the present focal point back to the desired location (namely the originally designed focal point on the central axis), the value of \( \beta \) needs to be redesigned to tune the \( y \) location of the present focal point. For example, when the waves are incident from the left bottom direction as illustrated in Figure 4.8(a), the present focal point is 'above' the desired one (indicated by the cross in the figure). Thus, the value of \( \beta \) needs to be decreased to move the present focal point 'down' to the central axis, as illustrated in figures from 4.8(b) to 4.8(d). Consequently, the harvester could again yield improved power in a large frequency band after this tuning process (see Figure 4.9).
Part B. Harvesting energy from transient elastic waves using the piezo-lens and SSHI-based techniques

In Part A, we have demonstrated that the piezo-lens can significantly improve the harvested power from waves and make the harvesting system adaptive to environment changes. In this part, the piezo-lens is further combined with SSHI-based harvesters to improve the harvested energy from transient waves. To demonstrate the harvesting performances in such applications, the model in Figure 4.10 is studied. It contains a finite plate (clamped at the right end), a piezo-lens and a harvester placed at the designed focal point of the lens to yield energy from the focused waves.

4.4 Harvesters

The harvester is composed of a piezoelectric patch bonded on the upper surface of the plate and a connected energy extraction circuit. A standard DC device and two SSHI-based devices are analyzed in this part, their topologies and the control strategies of the SSHI-based devices in transient wave energy harvesting are introduced hereinafter.

The DC device involves a rectifier to convert the voltage of the transducer to a DC form, and a capacitance $C_s$ to store the harvested energy, as depicted in Figure 4.11(a). The rectifier is assumed to be perfect. Thus, when the absolute value of
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

(a) $\alpha = 0.6/\pi$, $\beta = 0$

(b) $\alpha = 0.6/\pi$, $\beta = -0.04$

(c) $\alpha = 0.6/\pi$, $\beta = -0.08$

(d) $\alpha = 0.6/\pi$, $\beta = -0.12$

Figure 4.8: Adjusting the location of the energy concentration zone when waves are incident from the left bottom direction with an angle equal to $20^\circ$. Circles with solid line represent the energy concentration zones; crosses indicate the originally designed focal points.

Figure 4.9: Gain ratios of harvested power at different frequencies with the original ($\alpha = 0.6/\pi$, $\beta = 0$) and adjusted ($\alpha = 0.6/\pi$, $\beta = -0.1$) piezo-lenses. Waves are incident from the left bottom direction with an angle equal to $20^\circ$ and $R = 1000 \, \Omega$. 

4.4. Harvesters

Figure 4.10: The harvesting system with piezo-lens.

$V_h(t)$ is larger than or equal to the output voltage $V_{Cs}(t)$, the rectifier conducts, the storage capacitance is charged. Under this circumstance, the harvesting circuit is governed by the equations below:

$$
\dot{V}_h(t) = -\frac{Q_h(t)}{C_s}, \quad V_{Cs}(t) = |V_h(t)|
$$

(4.6)

On the other hand, when the absolute value of $V_h(t)$ is smaller than the output voltage $V_{Cs}(t)$, the rectifier is blocked, no charge will flow to the storage capacitance, the piezoelectric element is under open-circuit condition:

$$
\dot{V}_h(t) = \frac{I_{eq}(t)}{C_h}, \quad \dot{V}_{Cs}(t) = 0
$$

(4.7)

in which, the exact expression of $I_{eq}$ is given in Section 4.5.

The SSHI-based harvesters are obtained by integrating a SSHI interface into the DC one shown in Figure 4.11(a). In the first SSHI-based device, the SSHI interface, consists of a switch $S$ and an inductor $L$, is in parallel to the piezoelectric patch as illustrated in Figure 4.11(b). It is called parallel SSHI-DC (P-SSHI-DC) device in this paper. The control law of the switch is similar to the one used in steady state cases. At most of the time, the switch is opened, the device works just like a standard DC one. The switch is triggered at the time $t_i$ when the current $I_{eq}$ is null (namely, the voltage of the piezoelectric patch is a local maximum or minimum). It is kept closed for a very short period of time, corresponding to a half of the period of the oscillating circuit: $\Delta t = \pi \sqrt{LC_h}$. An inversion quality factor $Q_I$ [139] is used to describe the energy loss mainly caused by the inductor in the SSHI interface (note that in some papers [147, 144], the voltage inversion coefficient $\gamma = e^{-\frac{\pi}{Q_I}}$ is used). In the simulations, this loss is taken into account by adding a resistive part $R_L = \frac{1}{Q_I} \sqrt{\frac{L}{C_h}}$ with the inductor. Thus the governing equations for the circuit

\[\dot{V}_h(t) = -\frac{Q_h(t)}{C_s}, \quad V_{Cs}(t) = |V_h(t)|\]

(4.6)
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

during the inversion process are:

\[ L \ddot{Q}_h(t) + R_l \dot{Q}_h(t) + \frac{Q_h(t)}{C_h} = 0 \]
\[ \dot{V}_h(t) = \frac{\dot{Q}_h(t)}{C_h}, \quad \dot{V}_{C_s}(t) = 0 \] (4.8)

The second SSHI-based device contains a SSHI interface in series with the piezoelectric patch, as shown in Figure 4.11(c). This device is called series SSHI-DC (S-SSHI-DC) device here. At most of the time, the piezoelectric element is under open-circuit condition. Different from the steady state cases [143, 140], the switch is closed when \( I_{eq} = 0 \) and \( |V_h(t_i)| \geq V_{C_s}(t_i) \), the duration is \( \Delta t = \pi \sqrt{L \frac{C_L}{C_h}} \). An inversion quality factor \( Q_I \) is also used to take into account the loss of the switch interface in this case, accordingly a resistive part \( R_l = \frac{1}{Q_I} \sqrt{L \frac{C_L}{C_h} + C_s} \) is added to the inductor. The inversion process is governed by the equations below:

\[ L \ddot{Q}_h(t) + R_l \dot{Q}_h(t) + \frac{Q_h(t)}{C_h C_s/(C_h + C_s)} = 0 \]
\[ \dot{V}_h(t) = \frac{\dot{Q}_h(t)}{C_h}, \quad \dot{V}_{i}(t) = -L \ddot{Q}_h(t) - R_l \dot{Q}_h(t) \]
\[ V_{C_s}(t) = |V_h(t) - V_i(t)| \] (4.9)

in which, \( V_i(t) \) is the voltage difference between the two ends of the inductor.

4.5 Reduced finite element model

Instead of solving the full finite element model in Equation (4.2), which has a large number of DOFs, a reduced model is used in this part to study the harvesting performances. For the sake of simplicity, let:

\[ H_{dV} = [H_{dL}, \ H_{dh}], \quad V = [V_L, \ V_h]^T \]
\[ Q = [Q_L, \ Q_h]^T, \quad C_{Lh} = \begin{bmatrix} C_L & 0 \\ 0 & C_h \end{bmatrix} \] (4.10)

Using these notations and Equation (4.2), the governing equations of the piezoelectric system in Figure 4.10 in time domain are written as:

\[ M_{dd} \ddot{d} + H_{dd} \dot{d} + H_{dV} V = F \] (4.11a)
\[ -H_{dV}^T \dot{d} + C_{Lh} \dot{V} = \dot{Q} \] (4.11b)

Equation (4.11b) and the following Equations (4.15b), (4.18b) represent the Kirchhoff’s current law which must be satisfied at the joints where circuits are connected with the piezoelectric patches.
The reduced model is obtained through a transformation between the displacement $d$ and a set of modal coordinates $\eta$:

$$d = \Phi \eta$$  \hspace{1cm} (4.12)

in which, $\Phi = [\phi_1, \phi_2, ..., \phi_m]$. $\phi_i$ is the $i$th natural mode of the piezoelectric system under short-circuit condition with specific homogeneous Dirichlet boundaries, it is obtained by solving the following eigenvalue problem:

$$(-\omega_i^2 M_{dd} + H_{dd})\phi_i = 0$$  \hspace{1cm} (4.13)

here, $\omega_i$ is the corresponding natural frequency. The modes are mass-normalized, resulting in:

$$\Phi^T M_{dd} \Phi = I, \quad \Phi^T H_{dd} \Phi = \Lambda = \text{diag}(\omega_i^2)$$  \hspace{1cm} (4.14)

Using Equations (4.12) and (4.14), the governing Equations (4.11) are represented in modal coordinates as:
\[ \ddot{\eta} + \Lambda \dot{\eta} + \Phi^T H_{dV} V = \Phi^T F \]  
\[ -H_{dV}^T \Phi \dot{\eta} + C_{Lh} \dot{V} = \dot{Q} \]  
\[(4.15a)\]
\[(4.15b)\]

Only the first \( m \) modes in modal matrix \( \Phi \) will be retained, and the number is much smaller than that of the system's DOFs. Thus, the number of equations in (4.15) is largely reduced.

However, the reduced model in Equations (4.15) can't accurately describe the piezoelectric behaviors of the system [107] since the truncation of the higher order modes will lead to a static reduction error [128], which can be represented as:

\[ T_{err} = T_f - T_r = H_{dV}^T (H_{dd}^{-1} - \Phi \Lambda^{-1} \Phi^T) H_{dV} \]  
\[(4.16)\]

Here, \( T_f, T_r = V^{-1}Q \), they are respectively obtained by using the full and reduced models when the system is free from mechanical excitations.

The elements in \( T_{err} \) have specific physical meanings. The diagonal element \( T_{err}(i, i) \) represents the error of the charges \( Q(i) \) flowing into the \( i^{th} \) piezoelectric patch when a static voltage \( V(i) \) is applied on it. On the other hand, the non-diagonal element \( T_{err}(i, j) \) is the error of the charges \( Q(j) \) flowing into the \( j^{th} \) piezoelectric patch when a static voltage \( V(i) \) is applied on the \( i^{th} \) patch.

The non-diagonal elements in \( T_{err} \) are much smaller than the diagonal ones, hence they are negligible. Therefore, the error introduced by the mode truncation can be significantly reduced by eliminating the diagonal terms in \( T_{err} \). This is done by using a modified capacitance matrix \( C_{Lh}^* \) in the corrected model:

\[ C_{Lh}^* = \begin{bmatrix} C_L^* & 0 \\ 0 & C_h^* \end{bmatrix} = diag(H_{dV}^T (H_{dd}^{-1} - \Phi \Lambda^{-1} \Phi^T) H_{dV}) + C_{Lh} \]  
\[(4.17)\]

\( C_{Lh}^* \) is still a diagonal matrix, the diagonal terms of \( C_L^* \) and \( C_h^* \) represent the modified blocked intrinsic capacitances of corresponding patches. The corrected reduced model is obtained by replacing the \( C_{Lh} \) in Equation (4.15b) with \( C_{Lh}^* \):

\[ \ddot{\eta} + \Lambda \dot{\eta} + \Phi^T H_{dV} V = \Phi^T F \]  
\[ -H_{dV}^T \Phi \dot{\eta} + C_{Lh}^* \dot{V} = \dot{Q} \]  
\[(4.18a)\]
\[(4.18b)\]

This corrected reduced model has very good accuracy and is much more efficient than the full one. Validation details of this model can be found in Appendix D.

In Section 4.6, the corrected reduced model is used to study the focusing and harvesting performances of the system in Figure 4.10. The structural damping is taken into account by introducing a constant viscous damping coefficient \( 2\xi \omega_i \) into each remained mode. The NC circuits are connected to the piezoelectric patches through the following relation:
\[ \dot{Q}_L(t) = -C_{neg} \dot{V}_L(t) \] (4.19)

The DC and SSHI-based circuits are implemented in the corrected reduced models through Equations (4.6) to (4.9). Note that, in the equivalent model of the piezoelectric patch for harvesting, the current source is \( I_{eq}(t) = H_T \dot{\Phi} \Phi \eta(t) \) and the blocked intrinsic capacitance is the modified one \( C^* \).

## 4.6 Numerical results

In the simulations, the dimensions of the piezo-lens and host plate are illustrated in Figure 4.10. The geometry parameters of the piezoelectric cells and piezoelectric patch for harvesting are given in Table 4.1. The plate is clamped at the right end side and a 10-period Hanning-windowed tone burst excitation centred at frequency 2000 Hz is applied at the left end of the plate to generate transient waves. The excitation is a line transverse force, its waveform and spectrum are illustrated in Figure 4.12. Modes that have natural frequencies smaller than five times of the maximum frequency of interest are remained in the reduced model; a small damping coefficient \( \xi = 0.001 \) is used to minimize the influence of the structural damping on the wave propagation. The classical Runge-Kutta method is used to solve the ODEs using a fixed time step equal to \( 1 \times 10^{-6} \) s.

![Figure 4.12: The applied tone burst excitation.](image)

The piezo-lens’ parameters are chosen as \( \alpha = \pi/0.6 \), \( \beta = 0 \), and the harvester is placed at the corresponding focal point, which is located on the \( y = 0 \) line (namely the central axis) with a distance of 0.06 m from the right boundary of the lens. (Note that the focal length is 0.3 m counting from the left boundary.)

### 4.6.1 Comparison between the standard DC and SSHI-based devices

The standard DC and SSHI-based harvesters have been well studied in steady state, however lack of knowledge of their applications in transient wave cases could be found. Thus first of all, the performances of the systems with these devices in
transient wave energy harvesting are studied. The piezo-lens is applied in these studies. To assess the harvesting performances, two metrics are used. The first one is the harvested energy, which indicates how much energy can be yielded when a transient wave package passes through the harvester, it is the maximum energy that stored in the storage capacitance of the harvester:

$$E_{\text{har}} = \max\left(\frac{1}{2}C_sV_c^2(t)\right)$$

(4.20)

The second one is the harvesting efficiency, it is defined as [157]:

$$\eta_E = \frac{E_{\text{har}}}{E_{\text{input}}}$$

(4.21)

in which, $E_{\text{input}}$ is the total input energy introduced by the excitation. The efficiency indicates how much a harvesting system can convert the input mechanical energy ($E_{\text{input}}$) into the output electric one ($E_{\text{har}}$). This metric is a useful criterion to compare the harvesting performances of systems with different configurations and/or under different operating conditions.

The performances of systems with different harvesters are compared in Figure 4.13. In the simulations in Figure 4.13 and those hereinafter, the inversion quality factor $Q_I$ of the SSSI-based device is equal to 3. $Q_I$ indicates the energy loss caused by the harvesting circuit, typically a larger value of it means less energy loss [139]. $Q_I$ mainly depends on the involved piezoelectric material, the switch and the inversion inductance, its real value can only be obtained experimentally, herein the value of it is chosen according to the results in [139]. From Figure 4.13 it can be seen that the harvested energy and efficiency strongly depend on the storage capacitance, and the optimal capacitances corresponding to the maximum harvested energy and the maximum efficiency are nearly the same for each device. In addition, when the devices are all working at optimal conditions, the two SSSI-based devices have almost equal performances, and they both harvest 2.6 times more energy than the DC device and also have 2.6 times better efficiency.

To gain more insights into the harvesting performances of the systems with DC or
4.6. Numerical results

SSHl-based devices, the typical waveforms and the converted power are compared in Figure 4.14. The meanings of $V_h$, $I_{eq}$ and $V_C$, are given in Figure 4.11; the converted power is the product of $V_h$ and $I_{eq}$, positive it represents the amount of power converted from mechanical to electrical, and negative it means the contrary. In the simulations in Figure 4.14, the storage capacitances are all set as $C_s = 30C_h$ to guarantee an acceptable performance for all the devices according to the results in Figure 4.13. To facilitate the comparison, the maximum $V_C$ in the DC case is normalized to unit, and all the other voltages are normalized according to this value; a similar normalization process is used to deal with the converted power.

When the transient wave package reaches the harvester, the charges begin to accumulate in the storage capacitance; the accumulation stops after the main part of the package passes through the harvester. The charges accumulated in the storage capacitance are converted from strain energy by the piezoelectric transducer, it can be seen that the SSHL-based devices can promote the conversion of power from mechanical to electrical part but suppress the contrary, thus they can harvest more energy than the DC device. It can also be observed that when the SSHL-based devices are used, the waveforms of $I_{eq}$ are distorted. $I_{eq}$ depends directly on the strains of the piezoelectric patches, which are caused by the waves in the media, the distortions indicate that the harvesting processes in those cases could have non-negligible influences on the wave propagation at the location where the harvester is mounted. Since the switch is triggered at the time when $I_{eq} = 0$, the distortions result in multiple inversions, which cause the fluctuation of the harvesting performance as revealed in Figure 4.13.

4.6.2 Effects of piezo-lens on transient waves and harvesting

The piezo-lens can focus harmonic waves near the designed focal point in a large frequency band, thus it is expected that the piezo-lens could also focus transient waves if the main frequency components of the waves are within the effective frequency band of the piezo-lens, and harvesting at the designed focal point can yield improved energy. This expectation is verified in this subsection.

According to the results in Chapter 3, the piezo-lens used here is effective from about 100 Hz to 8000 Hz. Thus the major frequency components of the waves generated by the excitation in Figure 4.12 are totally within the effective frequency band. Firstly to study the effect of the piezo-lens on transient waves, the harvester is removed from the system since its presence at the focal point will make the energy concentration effect less obvious. The instantaneous transverse displacement $w_f$ at the designed focal point and input power are shown in Figure 4.15(a). The results corresponding to the case without lens are also given as references, the maximum amplitudes of the transverse displacement and input power in this case are both normalized to unity. The piezo-lens will reflect a part of the incident waves and these reflected waves will interact with the excitation forces, thus the input power is a little bit reduced when the piezo-lens is active. Even though with this reduction of input power, the maximum amplitude of $w_f$ is improved. This result indicates that
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Figure 4.14: Typical waveforms and converted energy when DC and SSHI-based devices are used to harvest energy from transient waves.
4.6. Numerical results

(a) upper: normalized transverse displacement at the focal point; lower: normalized input powers

(b) transverse displacement of the plate without lens at $t = 3.14\ ms$.

(c) transverse displacement of the plate with piezo-lens at $t = 3.09\ ms$.

Figure 4.15: Focusing transient waves.

The transient waves are focused by the lens. To further verify the focusing effect, the transverse displacement of the host plate at the time corresponding to the maximum amplitude of $w_f$ is depicted in Figure 4.15(b) for the case without piezo-lens and in Figure 4.15(c) for the case with piezo-lens. From the comparison, it can be clearly observed that the transient waves are focused near the designed focal point.

Figure 4.16 compares the harvesting performances between the cases with and without piezo-lens. When the piezo-lens is used and due to the focusing effect demonstrated above, the maximum harvested energy is enhanced about 2.5 times no matter which kind of harvester is used. The input power is reduced when the piezo-lens is applied, consequently the maximum harvesting efficiency is improved about 3 times for each device as illustrated in Figure 4.16(b).

In the above studies, it is verified that the piezo-lens can enhance the harvested energy from transient waves when the harvester is located at the designed focal point. However in the studies in Figure 4.15 the harvester is removed from the
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Figure 4.16: Comparison of the performances between the cases with and without piezo-lens. −−−: with piezo-lens; −−−: without lens. \( Q_I = 3 \).

system, it is not clear that whether the placement of the harvester near the piezo-lens will have significant influence on the location where the waves are concentrated. This is important since if the influence is non-negligible, the designed focal point may not be the optimal place for harvesting. The energy concentration location only depends on the incident waves when the parameters of the piezo-lens are specified, thus if the harvester has negligible effect on the excitation source, it will not affect the incident waves and the consequent energy concentration location. To study the influence of the harvester on the transient source, the instantaneous input power is used as a criterion. Figure 4.17 shows the input power induced by the transient excitation when different harvesters are used in the harvesting system. In these studies, the storage capacitance for each harvester is chosen to make the output energy maximum. The reference input power in the figure refers to the case without any harvester, and the maximum of it is normalized to unity. It can be observed that the harvester do not affect the input power, accordingly they won’t influence the generated transient waves and the energy concentration location.

4.6.3 Energy balance

Since the NC circuits in the piezo-lens are active elements and they need to be powered, it is also important to consider the energy balance of the harvesting system. In the piezo-lens, the NC circuits need to stiffen the structure. To realize such effect, the circuits should be fully reactive [158], namely, they don’t dissipate any energy.
4.6. Numerical results

Figure 4.17: Comparison of the input power when different devices are used to harvest the transient waves, the storage capacitance for each device is optimal, the reference input power refers to the case without any harvester.

Figure 4.18: Comparison of the consumed and harvested energy in the harvesting system, the storage capacitance for each device is optimal, $Q_I = 3$.

Thus, the active energy consumed by the NC circuit in a time interval from $t_1$ to $t_2$ is expressed as:

$$E_{C_{neg}} = \int_{t_1}^{t_2} V_{neg}(t)I_{neg}(t) = \frac{1}{2} C_{neg}(V_{neg}^2(t_2) - V_{neg}^2(t_1))$$  \hspace{1cm} (4.22)

For the transient wave cases here, the consumed energy is estimated in the interval between the time when the transient wave package first arrives at the piezo-lens and the time when the package leaves the lens. Figure 4.18 compares the consumed and harvested energy in the cases with different devices. The storage capacitance for each device is optimal (namely the output energy is maximum); the harvested energy of the system with piezo-lens and DC device is normalized to unity. When the piezo-lens is applied in harvesting systems, it is observed that the amount of consumed energy is really small compared with the harvested one, 11% with the DC device and less than 4% with the SSHI-based devices. Thus even though we take into account the energy consumed by the piezo-lens, the harvesting systems incorporating the piezo-lens still can yield considerably improved energy compared with the cases without piezo-lens.
4.6.4 Practical application considerations

In previous sections, performance of the harvesting systems are verified in ideal situations with very small structural damping ($\xi = 0.001$) and neglected forward voltages of diodes. In this section, influences of these factors on harvesting performance are considered, and applicability of the harvesting systems is discussed. In new examples, the structural damping is given as $\xi = 0.05$, the forward voltage of each diode is 0.6 V, the inversion quality factor $Q_I$ for SSHI interface is chosen as 5.6 according to the experimental results in [139], and the storage capacitance $C_s$ for each harvester is chosen to make the output energy maximum, i.e., $C_s = 14C_h^*$ for the DC harvester, $C_s = 7.4C_h^*$ for the P-SSHI-DC harvester and $C_s = 30C_h^*$ for the S-SSHI-DC harvester ($C_h^* = 22.0 \text{ nF}$).

Figure 4.19 shows the waveforms of displacements and voltages in harvesting systems incorporating a piezo-lens. The measure location of the displacement is the left-bottom corner on the upper surface of the piezoelectric patch for harvesting. It can be observed that even though the maximum amplitude of the mechanical response of the patch is only micrometer-scale, the harvesting systems are adequate to yield energy from the transient waves. It is noted here that the fast switch actions illustrated in Figure 4.19(b) and 4.19(c) perhaps are difficult to obtain in piezoelectric mechanical systems. However, the piezo-lens can work at lower frequencies as long as the wavelengths are smaller than the characteristic length of the lens [155]. In those cases, the switch actions could be more realistic.

The improvement of harvesting performance by using piezo-lens in these new cases are illustrated in Figure 4.20, in which, the mean power $P_1$ for the case without lens and the mean net power $P_2$ for the case with piezo-lens are defined as:

$$P_1 = \frac{E_{har1}}{t_{har1}}, \quad P_2 = \frac{E_{har2} - E_{C_{neg}}}{t_{har2}}$$  \hspace{1cm} (4.23)

Here, $E_{har1}$ and $E_{har2}$ are the harvested energy obtained by Equation (4.20), $E_{C_{neg}}$ is the consumed energy by the piezo-lens evaluated by Equation (4.22), $t_{har1}$ and $t_{har2}$ are the charging duration of the storage capacitances (see Figure 4.19). From Figure 4.20, it can be seen that the piezo-lens is also very effective to improve the harvesting performance in these more realistic cases.

Indeed, to realize the harvesting systems, there are still challenges. The NC circuits required in the piezo-lens need to be totally reactive to stiffen the structure, but the existing circuits that could realize negative capacitance all contain resistive parts which more or less will dissipate some energy [159, 100], thus they are potentially not suitable to realize a piezo-lens. However, it is hopeful that approximate fully reactive NC circuits could be achieved to reach the required stiffening effect in the coming future by optimizing the existing circuits (see [159]) or using synthetic circuits as proposed in [160]. Since the NC circuits for the piezo-lens are not determined at present, giving a rigorous energy balance analysis which takes into account the dissipated power is impossible, but it is logical to predict that the dissipated power by the piezo-lens could be very low since the NC circuits need to be
4.6. Numerical results

(a) standard DC, $C_s = 14C_h$

(b) P-SSHI-DC, $Q_I = 5.6$, $C_s = 7.4C_h$

(c) S-SSHI-DC, $Q_I = 5.6$, $C_s = 30C_h$

Figure 4.19: Waveforms of transverse displacement and voltages when DC and SSHI-based devices are used to harvest energy from transient waves. The measure location of the displacement is the left-bottom corner on the upper surface of the piezoelectric patch for harvesting.
Chapter 4. Enhancement of elastic wave energy harvesting using adaptive piezo-lens

Figure 4.20: Comparison of the mean harvested power between the cases without and with piezo-lens.

(approximately) fully reactive. Besides, in some applications, the energy balance is not a critical issue. For example, a potential application of the harvesting systems is in Structural Health Monitoring (SHM) in large-scale structures. Sensors are sometimes embedded in structures at different locations in those cases, the maintenance is very difficult or even impossible when traditional batteries are used to power the sensors. Thus, the harvesting systems can be used to power these sensors. In these applications, the piezo-lens can be placed on the surfaces of structures thus it can be powered by traditional batteries rather than harvesters.

4.7 Conclusions of Part A and B

In this chapter, the piezo-lens is explored to improve the harvested power from traveling waves. Part A discussed the concept and effect of using piezo-lens in wave energy harvesting. The piezo-lens can focus waves near a designed focal point, consequently forming a zone with concentrated energy. This energy concentration effect is observed in a large frequency band, from about 100 Hz to 8000 Hz. Due to this broadband energy concentration effect, when a harvester is placed at the designed focal point, the harvested power is considerably improved within the effective frequency band of the lens. Except the energy concentration effect, the piezo-lens is also adaptable. By tuning the NC values of the cells composing the piezo-lens, the energy can be focused at different locations. The focal point is recommended to be tuned on the central axis of the lens and near the lens boundary to obtain better energy concentration effect. In addition, the tunable feature can be exploited to handle the change of wave incident direction. The piezo-lens is initially designed to focus waves from the Ox direction. In practice, the waves’ direction may be changed. Under these circumstances, the energy is concentrated away from the designed focal point, at where the harvester is placed. Consequently, the harvested power is significantly reduced. However, by tuning the piezo-lens, the energy concentration zone can be adjusted back to the desired location to improve the harvested power.

In Part B, the piezo-lens is combined with SSHI-based harvesters to obtain enhanced energy from transient waves. The standard DC, parallel SSHI-based and
series SSHI-based harvesters are used. The SSHI interface can promote the conversion of the power from mechanical to electric part but suppress the contrary, accordingly the SSHI-based harvesters are more efficient than the standard DC one for harvesting energy from transient waves. The piezo-lens can focus transient waves near a designed focal point, thus placing the harvesters at that point can enhance the harvested energy about 2.5 - 3 times and improve the harvesting efficiency about 3 times as compared with the cases without lens. This improvement of harvesting performance is obtainable when the realistic damping effect of the host structure and the forward voltages of diodes are taken into account.
Start from this chapter, the second type of adaptive GRadient INdex (GRIN) structure, namely the time-space modulated structure is studied. Longitudinal waves in time-space modulated beams are considered to study the properties of Bloch modes in time-space modulated structures and the frequency conversion induced by them. This chapter is organized as follows. Section 5.1 introduces the time-space modulated beam model. Section 5.2 presents the Plane Wave Expansion (PWE) method we used to obtain the Bloch modes and studies the properties of the Bloch modes. Section 5.3 demonstrates and explains two types of frequency conversion induced by time-space modulated beams. Conclusions are given in Section 5.4.

5.1 Time-space modulated beam

The beam lying along the $x$ axis in Figure 5.1(a) is studied. The density of the beam $\rho_0$ is constant and homogeneous, while the Young’s modulus is modulated in time and space according to a cosine wave function:

$$E(x, t) = E_0 + E_m \cos(\omega_m t - k_m x)$$  \hspace{1cm} (5.1)
in which, $E_0$ is the Young’s modulus when there is no modulation, $E_m$ is the modulation amplitude, $\omega_m$ and $k_m$ are respectively the angular frequency and wavenumber of the modulation wave, whose wavelength is $\lambda_m = 2\pi/k_m$. The modulation wave propagates along the beam with the speed $v_m = \omega_m/k_m$, as illustrated in Figure 5.1(b). In the following discussion, the modulation expressed in Equation (5.1) is described by two dimensionless parameters, namely the dimensionless modulation amplitude $\alpha_m = E_m/E_0$ and dimensionless modulation speed $\beta_m = v_m/c_0$, here $c_0 = \sqrt{E_0/\rho_0}$ is the phase velocity of the longitudinal wave in an uniform beam. The modulation wave could propagate in both direction, $\beta_m > 0$ indicates the wave propagates in the positive direction and $\beta_m < 0$ means the opposite. Note that the time-space periodicity can also be obtained by modulating the density, or the Young’s modulus and density simultaneously [94]. The modulation wave is not limited to harmonics, it can also be square or triangular waves [93].

5.2 Dispersion relations and Bloch modes in time-space modulated beams

5.2.1 Analysis of dispersion

The longitudinal motion $u(x,t)$ in the modulated beam under the long-wavelength limit is governed by:

$$\frac{\partial}{\partial x} \left[ E(x,t) \frac{\partial u(x,t)}{\partial x} \right] - \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

(5.2)
5.2. Dispersion relations and Bloch modes in time-space modulated beams

Our studies are restricted to the cases satisfying following condition [79]:

$$|\beta_m| < \frac{1}{\sqrt{1 + \alpha_m}}$$

(5.3)

under which, the time-space system is stable and the solution of Equation (5.2) can be expressed in the generalized Bloch mode form:

$$u(x, t) = \sum_{q=-\infty}^{+\infty} U_q e^{i[(\omega + q\omega_m)t - (k + qk_m)x]}$$

(5.4)

Following the PWE method [93] to obtain the dispersion relations, the Young’s modulus function in Equation (5.1) is expanded by using Fourier series:

$$E(x, t) = \sum_{p=-\infty}^{+\infty} \hat{E}_p e^{ip(\omega_m t - k_m x)}$$

(5.5)

in which, $\hat{E}_p$ are the Fourier coefficients of corresponding bases.

Substituting equations (5.4) and (5.5) into (5.2) and forming the complex inner product with $e^{in(\omega_m t - k_m x)}$ (its complex conjugate appears in the integral), exploiting the orthogonality of the Fourier basis that only the inner products who satisfy $q + p = n$ are nonzero yields:

$$\sum_{q=-\infty}^{\infty} (k + qk_m)(k + nk_m)\hat{E}_{n-q}U_q = (\omega + n\omega_m)^2 \rho_0 U_n$$

(5.6)

By truncating the sum in Equation (5.4) to a finite number $2N + 1$ of terms, and evaluating Equation (5.6) for $-N \leq n \leq N$, a well-posed Quadratic Eigenvalue Problem (QEP) is formed with a set of $2N + 1$ equations. The QEP could be solved in terms of $k$ for a given $\omega$ and vice versa. The QEP results in $4N + 2$ eigenvalues and $4N + 2$ corresponding eigenvectors, each of size $2N + 1$. Each eigenvalue together with the corresponding eigenvalue in the form of Equation (5.4) represents a Bloch mode that could exist in the structure. Therefore, there are $4N + 2$ longitudinal Bloch modes solved from the QEP.

5.2.2 Band diagrams

The dispersion relations are illustrated by representing dispersion curves on a Brillouin type of diagram of frequency vs wavenumber. To make the analysis more general, the following dimensionless frequency $\Omega$ and dimensionless wavenumber $\mu$ are used:

$$\Omega = \frac{\lambda_m \omega}{2\pi\alpha_0}, \quad \mu = \lambda_m k$$

(5.7)

Figure 5.2 shows the dispersion relations of longitudinal modes in uniform, space-only periodic and time-space modulated beams. In uniform beams ($\alpha_m = \beta_m = \ldots$)
0), only the fundamental harmonics can exist, thus in Figure 5.2(a), only the two branches departing from the origin are illustrated, one of them is associated to a positive-going wave and the other is associated to a negative-going one. In space-only periodic beams (\( \alpha_m \neq 0, \beta_m = 0 \)), higher order Bloch modes occur, thus their band diagrams are composed of branches corresponding to different orders Bloch modes, as shown in Figure 5.2(b). Bloch modes in space-only periodic beams are not unique, i.e., all Bloch modes that could exist in the beams occur for wavenumber \( \mu \)-values within the First Brillouin Zone (FBZ), where \( \mu \) varies in \([-\pi, \pi]\). This property could be directly observed from the expansion in Equation (5.4). When the wavenumber is shifted by \( n k_m \) (in which \( n \) is an integer), the expansion:

\[
0(x, t) = e^{i(\omega t - kx)} \sum_{q=-N}^{N} U_q e^{-i(qk_m)x} = e^{i[\omega_t - (k + nk_m)x]} \sum_{q=-N}^{N} U_q e^{-i(q-n)k_m x} \quad (5.8)
\]

is left unchanged but for a shift in Bloch mode order from 0 (indicating the 0th Bloch mode) to \( n \) (indicating the \( n \)th Bloch mode). Due to this non-uniqueness, the dispersion curves repeat with period \( 2\pi \alpha_m \) along the wavenumber axis (band folding), thus dispersion relations can be simply illustrated in the FBZ [161, 3]. The uniform beams present no band gap as shown in Figure 5.2(a), on the contrary, when space periodicity is applied, Bragg-type of stop bands are created at edges of Brillouin zones (i.e., at \( \mu = n\pi, n = 0, \pm 1, \ldots, \pm N \)), as indicated by shadows in Figure 5.2(b). Moreover, wave propagation in uniform and space-only periodic beams (also in other conventional structures) obeys reciprocity, that is if a wave can propagate from point A to point B in the structure, it is also possible to propagate oppositely from B to A in an equal manner. The reciprocity also be understood as that if a wave with wavenumber \( \omega(\mu) \) can exist in a structure, a wave with \( \omega(-\mu) \) will also exist in this structure, i.e., the band diagram is symmetry with respect to the line \( \mu = 0 \) (see Figures 5.2(a) and 5.2(b)).

When time-space modulation is introduced into the beam (\( \alpha_m \neq 0, \beta_m \neq 0 \)), the non-uniqueness of Bloch modes is still presented but in a different way compared with the space-only modulation case. It can be seen that a shift in frequency of \( n\omega_m \) accompanied by a shift in wavenumber of \( nk_m \), as follows:

\[
0(x, t) = e^{i(\omega t - kx)} \sum_{q=-N}^{N} U_q e^{iq(\omega_m t - k_m x)}
\]

\[
= e^{i[(\omega + n\omega_m)t - (k + nk_m)x]} \sum_{q=-N}^{N} U_q e^{i(q-n)(\omega_m t - k_m x)} \quad (5.9)
\]

also results in invariant of the expansion but a shift of the Bloch mode order. Thus, dispersion curves in the time-space modulation case repeat along an oblique line with the slope \( \Omega_m/\mu_m = \beta_m/2\pi \) (\( \Omega_m \) and \( \mu_m \) are the dimensionless modulation frequency and wavenumber, respectively) rather than along the wavenumber axis,
5.2. Dispersion relations and Bloch modes in time-space modulated beams

Figure 5.2: Band diagrams of longitudinal modes in beams: (a) uniform beams with $\alpha_m = \beta_m = 0$, (b) space-only periodic beams with $\alpha_m = 0.4$ and $\beta_m = 0$, (c) time-space periodic beams with $\alpha_m = 0.4$ and $\beta_m = 0.2$, (d) time-space periodic beams with $\alpha_m = 0.4$ and $\beta_m = -0.2$. Shadows indicate stop bands.

as shown in Figures 5.2(c) and 5.2(d). Another significant influence of the time-space modulation on dispersion relations is that the symmetry of the band diagram (with respect to line $\mu = 0$) is broken. Therefore, it is impossible to find two equal modes propagating in two opposite directions, i.e., wave propagation in time-space modulated structures is non-reciprocal.

Although an extra periodicity in time domain exists, Bragg-type of stop bands are still created in time-space modulated media [79, 93]. It can be seen from Figures 5.2(c) and 5.2(d) that, stop bands of Bloch modes are separated. The relative locations of these stop bands only depend on the modulation velocity $\beta_m$, larger absolute value of $\beta_m$ results in farther distance between them in terms of frequency, as shown in Figure 5.3. For modulation velocities with absolute values smaller than a critical value (which is $\alpha_m/4$ given by [93]), the adjacent two stop bands will
have some overlap (Figure 5.3(a)); for modulation velocities with absolute values larger than the critical value but smaller than $\frac{1}{\sqrt{1+\alpha_m}}$ (see conditions of stability in (5.3)), all stop bands are well separated (Figures 5.3(b) and 5.3(c)). Note that, the dispersion curves of modulated beams with negative $\beta_m$ values can be obtained by flipping in the horizontal direction the corresponding band diagrams of beams with positive $\beta_m$ values (see Figures 5.2(c) and 5.2(d)). Therefore, those results are not shown.

![Dispersion Curves](image)

(a) $\alpha_m = 0.4, \beta_m = 0.05$  
(b) $\alpha_m = 0.4, \beta_m = 0.4$  
(c) $\alpha_m = 0.4, \beta_m = 0.6$

Figure 5.3: Influences of the modulation velocity $\beta_m$ on the dispersion curves.

In the above analyses, the dispersion curves are obtained by solving the QEP in terms of $\omega$ for given $k$ in a range of interest. In this way, the stop bands are represented by gaps in the band diagram, therefore they are easy to be observed. However, the drawback is that it is difficult to distinguish and classify the Bloch modes, which is important for the following studies. Accordingly, in what follows, we use an alternative way to solve the QEP by fixing $\omega$ and seeking $k$. Figure 5.4 shows the obtained dispersion curves of modes in a modulated beam ($\alpha_m = 0.4$, $\beta_m = 0.2$) illustrated on a diagram of wavenumber vs frequency. In that figure, the
5.2. Dispersion relations and Bloch modes in time-space modulated beams

Bloch modes are classified into positive-going and negative-going groups according to their group velocities, which are calculated by \( c_g = \partial \omega / \partial k \). Bloch modes in each group are organized according to their wavenumber in an ascending fashion. The \( n \)th Bloch modes in the positive-going and negative-going groups are respectively represented by \( u^+_n \) and \( u^-_n \), with \( n = -N,...,0,...,+N \). According to Equation (5.4), they are expressed as a group of harmonics:

\[
\begin{align*}
    u^+_n(x, t, k^+_n, \omega) &= \sum_{q=-N}^{+N} U^+_{(n,q)} e^{i[(\omega + q\omega_m)t - (k^+_n + qk_m)x]} \\
    u^-_n(x, t, k^-_n, \omega) &= \sum_{q=-N}^{+N} U^-_{(n,q)} e^{i[(\omega + q\omega_m)t - (k^-_n + qk_m)x]}
\end{align*}
\]

(5.10)

in which, \( k^\pm_n \) are eigenvalues obtained from the QEP, \( \{U^+_{(n,-N)}, ..., U^+_{(n,0)}, ..., U^+_{(n,+N)}\}^T \) and \( \{U^-_{(n,-N)}, ..., U^-_{(n,0)}, ..., U^-_{(n,+N)}\}^T \) are the corresponding eigenvectors.

In Figure 5.4, the frequency ranges where the real part of \( \mu \) is constant are stop bands of corresponding Bloch modes. It can be seen that, stop bands of different orders Bloch modes occupy different frequency ranges. In some previous works [93, 94], the two stop bands of \( u^+_0/u^-_1 \) and \( u^-_0/u^+_1 \) modes in Figure 5.4 are termed directional band gaps, i.e., within these stop bands only positive-going or negative-going modes can propagate. It is remarked here that these descriptions are not appropriate. Because, for example, at frequencies within the stop bands of \( u^-_0/u^+_1 \) modes, even though these two modes can’t propagate in the beam, the other orders positive and negative-going Bloch modes can. Therefore, all these Bloch modes must be taken into account when studying this type of time-space modulated systems.

5.2.3 Bloch modes as group of harmonics

Equations (5.10) show that Bloch modes can be further expressed as superposition of a series of harmonic components. The amplitudes of the \( q \)th harmonic components of the \( n \)th positive and negative-going Bloch modes are respectively \( |U^+_{(n,q)}| \) and \( |U^-_{(n,q)}| \), they phase velocities are defined as \( c^\pm_{(n,q)} = \omega^\pm_{(n,q)} / k^\pm_{(n,q)} \). Note that if \( k^\pm_n \) are complex, only the real parts of them are used to obtain the phase velocities.

To study the properties of harmonics composing the Bloch modes, harmonic amplitudes and phase velocities of the \( u^+_0 \) Bloch mode in a time-space modulated beam at different frequencies are illustrated in Figure 5.5(c) and 5.5(d). Those in a space-only modulated beam are also shown in Figure 5.5(a) and 5.5(b) as references. In these figures, \( |U^+_{(0,q)}| \) and \( c^+_{(0,q)} \) respectively represent the amplitude and phase velocity of the \( q \)th harmonic. Besides, these amplitudes and phase velocities are normalized by their maximum absolute values, respectively. In order to illustrate both positive and negative phase velocities on a same logarithmic plot, the sign of the negative phase velocity has been reversed. Also in Figure 5.5(d), the phase velocity \( c_0 \) of the longitudinal wave in a uniform beam is shown.
It can be seen from Figures 5.5(a) and 5.5(c) that in both space-only and time-space modulation cases, at frequencies far from the stop bands, the $u_0^+$ modes are dominated by the 0\textsuperscript{th} harmonic. In the neighborhood and inside the stop bands, contributions of two components are significant, they are the 0\textsuperscript{th} and $-1\textsuperscript{st}$ harmonics. It is also noticeable from Figures 5.5(b) and 5.5(d) that there are negative-going harmonics. The existence of negative-going components in a positive-going Bloch mode is induced by reflections inside the periodic beam \cite{162}, which must occur at any section of the beam due to the continuous periodic variation of the impedance caused by the modulation of Young’s modulus.

With regard to the detailed behavior of the harmonic amplitudes and phase velocities, there are significant differences between the space-only and time-space modulation cases. From Figures 5.5(a) and 5.5(b) it can be seen that in the case of space-only period structures, at stop band frequencies (including the two band edges), there is a matching, in pairs, of positive and negative-going harmonics in amplitude and phase velocity, e.g., the 0\textsuperscript{th} and $-1\textsuperscript{st}$, the 1\textsuperscript{st} and $-2\textsuperscript{nd}$. Thus the Bloch mode, at those frequencies, is composed of pairs of equal and opposite-going harmonics, leading to zero flux of energy along the beam. However, in the time-space modulated case, no such matching can be observed in Figures 5.5(c) and 5.5(d). This is because the Bloch mode in time-space modulation cases consists of harmonics having different frequencies and wavenumber (see Equations (5.10)), to satisfy the zero power flux condition within the stop band and at band edges, these harmonics must have different amplitudes and phase velocities. From Figures 5.5(b) and 5.5(d), it can also be seen that the phase velocities of harmonics except...
5.3. Frequency conversion

Figure 5.5: Amplitudes and phase velocities of harmonic components of the $u_0^+$ Bloch mode vs frequency. (a), (b): in a space-only periodic beam; (c), (d): in a time-space periodic beam. Shadows indicate stop bands.

The 0th one are zero at 0 Hz in space-only modulation case but nonzero in time-space modulation case. These phase velocities in time-space modulation case are all equal to $\beta_m c_0$ at 0 Hz, which is 0.2$c_0$ in the presented study. This means that, the time-space modulation gives these harmonics an initial velocity equal to the modulation velocity. As a consequence, the full-frequency-range negative-going harmonics in space-only modulation case (e.g., the $-1^{st}$, $-2^{nd}$ and $-3^{rd}$ harmonics) present a direction inverse process in time-space modulation case as the frequency increases. The inverse frequency of the $q^{th}$ harmonic is $|q| \cdot \beta_m$. For example, in Figure 5.5(d), the inverse frequency of the $-1^{st}$ harmonic is 0.2 and that of the $-2^{nd}$ harmonic is 0.4.

5.3 Frequency conversion

In what follows, we demonstrate and explain two types of frequency conversion induced by time-space modulated structures. The first type can be observed when waves inside the modulated structures are reflected by insulating boundaries of the structures; the second can be observed when external waves are reflected by interfaces between homogeneous and modulated structures.
5.3.1 Theory

5.3.1.1 Reflection at ends of time-space modulated beams

We consider the semi-infinite time-space modulated beam occupying the region \(-\infty < x \leq 0\), as shown in Figure 5.6. The beam end at \(x = 0\) is free. Note that one can also apply other kinds of boundary conditions, corresponding results can be obtained using the same process introduced hereinafter.

Assume that waves \(u_i(x,t)\) composed of only \(u_0^+ (\omega)\) mode are incident on the end:

\[
u_i(x, t) = u_0^+ = \sum_{q=-N}^{+N} U_0^+ e^{i[\omega + q\omega_m]t - (k_0^+ + qk_m)x]}
\]  

(5.11)

The reflected waves \(u_r(x,t)\) are represented as superposition of the negative-going Bloch modes (Equations (5.10)) supported by the modulated beam:

\[
u_r(x, t) = \sum_{n=-N}^{+N} B_n u_n^- = \sum_{n=-N}^{+N} \sum_{q=-N}^{+N} B_n U_{n,q}^- e^{i[\omega + q\omega_m]t - (k_n^- + qk_m)x]}
\]  

(5.12)

\(B_n\) are contribution coefficients of corresponding Bloch modes.

These waves are constrained by the conservation of momentum at the end:

\[
E(0, t) \frac{\partial [u_i(x,t) + u_r(x,t)]}{\partial x} \bigg|_{x=0} = 0
\]  

(5.13)

Substituting the expressions of waves (Equation (5.11) and (5.12)) into Equation (5.13) results in:

\[
\sum_{q=-N}^{+N} \{(k_0^+ + qk_m)U_0^+ + \sum_{n=-N}^{+N} [B_n(k_n^- + qk_m)U_{n,q}^-] \} e^{i(\omega + q\omega_m)t} = 0
\]  

(5.14)

By exploiting the orthogonality of harmonic functions \(e^{i(\omega + q\omega_m)t}\), a set of linear equations are obtained from Equation (5.14). Solving these equations, we can obtain the coefficients \(B_n\).
5.3. Frequency conversion

\[ B = -M_2^{-1}M_1 \]  
(5.15)

in which, \( B = \{ B_{-N}, ..., B_0, ..., B_N \}^T \), \( M_1(q + N + 1) = (k_0^+ + q k_m)U_{(0,q)}^+ \) with \( q = -N, ..., 0, ..., N \) and \( M_2(q + N + 1, n + N + 1) = (k_n^- + q k_m)U_{(n,q)}^- \) with \( n, q = -N, ..., 0, ..., N \).

5.3.1.2 Reflection and transmission at interfaces between homogeneous and time-space modulated beams

Consider that a semi-infinite homogeneous beam (occupying \(-\infty < x < 0\)) is connected with a semi-infinite modulated beam (occupying \(0 \leq x < +\infty\)) at \(x = 0\), as shown in Figure 5.7. The Young’s moduli of these two parts are \( E_0 \) and \( E(x, t) \) (Equation (5.1)), respectively.

![Figure 5.7: Reflection and transmission at the interface between a homogeneous and a time-space modulated beam.](image)

Assume that a single harmonic is incident on the modulated beam from the left side:

\[ u_i(x,t) = e^{i \omega (t - \frac{x}{c_0})} \]  
(5.16)

The induced waves \( u_t(x,t) \) in the modulated beam are represented as superposition of the positive-going Bloch modes (Equations (5.10)):

\[ u_t(x,t) = \sum_{n=-M}^{+M} T_n u_n^+ = \sum_{n=-M}^{+M} \sum_{q=-M}^{+M} T_n U_{(n,q)}^+ e^{i(\omega + q \omega_m)(t-(k_n^+ + q k_m) x)} \]  
(5.17)

here, \( T_n \) are contribution coefficients of corresponding Bloch modes, \( M \) is the truncation order of Bloch modes in \( u_t(x,t) \), constraint between it and the truncation order \( N \) of the QEP will be discussed later.

It can be seen from Equation (5.17) that the induced waves inside the modulated beam have harmonic components with frequencies \( \omega + q \omega_m \), \( q = -M, ..., 0, ..., M \). Therefore, the reflected waves \( u_r \) must be superposition of harmonics be of all possible frequencies \( \omega + q \omega_m \):

\[ u_r(x,t) = \sum_{q=-M}^{+M} R_q e^{i(\omega + q \omega_m)(t + \frac{x}{c_0})} \]  
(5.18)
$R_q$ are amplitudes of corresponding harmonics.

These waves are constrained by the continuity of displacement and conservation of momentum at the interface:

$$u_i(0, t) + u_e(0, t) = u_t(0, t)$$
$$E_0 \frac{\partial [u_i(x, t) + u_e(x, t)]}{\partial x} \bigg|_{x=0} = E(0, t) \frac{\partial u_t(x, t)}{\partial x} \bigg|_{x=0}$$

(5.19)

Substituting the expressions of waves (equations (5.16) to (5.18)) and the 2D Fourier expansion of the Young’s modulus of the modulated beam (Equation (5.5)) into the above continuity conditions:

$$e^{i\omega t} + \sum_{q=-M}^{+M} \{R_q - \sum_{n=-M}^{+M} T_n U_{(n,q)}^+ \} e^{i(\omega + q\omega_m) t} = 0$$

(5.20)

$$- E_0 \frac{\omega}{c_0} e^{i\omega t} + \sum_{q=-M}^{+M} \{ E_0 \frac{\omega}{c_0} q\omega_m R_q$$

$$+ \sum_{n=-M}^{+M} \sum_{n=-M}^{+1} \hat{E}_p T_n (k_n^+ + (q-p)k_m) U_{(n,q-p)}^+ \} e^{i(\omega + q\omega_m) t} = 0$$

Recall that $\hat{E}_p (p = -1, 0, 1)$ are the Fourier coefficients in the Fourier expansion of the modulated Young’s modulus. $U_{(n,q-p)}$ in the second equation in Equation (5.20) are elements in eigenvector $\{ U_{(n,-N)}^+, ..., U_{(n,0)}^+, ..., U_{(n,+N)}^+ \}^T$. Therefore the integral index $q - p$ must satisfy $-N \leq q - p \leq +N$. For indexes $q$ and $p$, respectively, they satisfy $-M \leq q \leq M$ and $-1 \leq p \leq 1$. Taking all these conditions into account, we have $M \leq N - 1$. That’s the constraint between $M$ and $N$ mentioned in the above.

Again by exploiting the orthogonality of harmonic functions $e^{i(\omega + q\omega_m) t}$, the two equations in (5.20) can be rewritten into the following matrix forms:

$$R - M_3 T = -I_1$$
$$M_1 R + M_2 T = M_6 I_1$$

(5.21)

here, $R$ and $T$ are column vectors respectively containing coefficients $R_q$ and $T_n (n, q = -M, ..., 0, ... M)$, $I_1$ is a $(2M + 1)$-by-1 vector, in which, $I_1 (M + 1) = 1$ and other elements are zero, matrices $M_i$ are all $(2M + 1)$-by-$(2M + 1)$, details of them are:

$$M_3(q + M + 1, n + M + 1) = U_{(n,q)}^+$$
$$M_4 = diag(E_0(\omega + q\omega_m)/c_0)$$

$$M_5(q + M + 1, n + M + 1) = \sum_{p=-1}^{+1} \hat{E}_p (k_n^+ + (q-p)k_m) U_{(n,q-p)}^+$$
$$M_6 = E_0 \frac{\omega}{c_0} I$$

(5.22)
5.3. Frequency conversion in which, \( n, q = -M, \ldots, 0, \ldots, M \), \( I \) is an identity matrix.

Solving Equation (5.21), we obtain the coefficients \( R \) and \( T \). Then using expressions Equation (5.17) and (5.18) we have the induced waves inside the modulated beam and the reflected waves.

### 5.3.2 Results

#### 5.3.2.1 Frequency conversion at ends of time-space modulated beams

In this section, we analyze the frequency conversion induced by reflection at ends of time-space modulated beams using the theory in Section 5.3.1.1. The truncation order used in our simulations is \( N = 4 \), as will be seen it is large enough to take into account all considerable harmonics. We assume that the incident wave \( u_i(x,t) \) is from a source at \( x_1 = -10 \lambda m \), as shown in Figure 5.6. We concern the propagating harmonics among the reflected waves. Therefore, we study the harmonic amplitudes of the incident and reflected waves at \( x_1 \), at where evanescent waves generated at the end are already significantly decayed. According to Equation (5.11) and (5.12), at \( x_1 \) the \( q^{th} \) harmonic amplitudes of the incident and reflected waves are

\[
|U_{(0,q)}^+ e^{-i(k_0^+ + q \beta_m) x_1}| \quad \text{and} \quad |\sum_{n=-N}^{+N} B_n U_{(n,q)}^- e^{-i(k_0^- + q \beta_m) x_1}|,
\]

respectively. Figure 5.8 shows the considerable harmonic amplitudes in two cases. Figure 5.8(a) and (b) show the results of the first case, in which, the incident and modulation waves all travel in the positive direction \( (\alpha_m = 0.4, \beta_m = 0.2) \). Figure 5.8(c) and (d) show the results of the second case, the incident wave is positive-going but the modulation wave is negative-going in this case \( (\alpha_m = 0.4, \beta_m = -0.2) \). In both cases, amplitudes are normalized by the corresponding amplitude \( |U_{(0,0)}^+ e^{-i k_0^+ x_1}| \) of the 0th harmonic of the incident wave. Note that the dimensionless frequency \( \Omega = \lambda m \omega / (2\pi c_0) \) is used in Figure 5.8 and in what follows.

When the incident wave and the modulation wave have the same direction, from Figure 5.8(a) and (b) we can see that, outside the two stop bands, normally, the dominant harmonic of the reflected wave is coincident with that of the incident wave. Inside the stop band of \( u_0^+ \) mode, the incident harmonics are evanescent, they decay rapidly toward the end. Therefore, reflected harmonic amplitudes at these frequencies are small. Inside the stop band of \( u_0^- \) mode, the incident wave is dominated by the 0th harmonic. However, the reflected wave is dominated by the 1st one, frequency of which is \( \Omega + \beta_m \) (because the dimensionless modulation frequency is \( \Omega_m = \lambda m \omega_m / (2\pi c_0) = \beta_m \)), inside the stop band of \( u_0^+ \) mode (the difference between two corresponding frequencies respectively in the stop bands of \( u_0^- \) and \( u_0^+ \) modes is \( \beta_m \), as indicated in Figure 5.8(a)). Therefore, the main frequency is upconverted from the stop band of \( u_0^- \) mode to that of \( u_0^+ \) mode after the reflection. There is an obvious exceptional sharp peak of the amplitude of the \(-2^{nd}\) harmonic at \( \Omega = 0.4 \). This peak is caused by the rigid body motion of the beam, because frequency \( (\Omega - 2\beta_m) \) of the \(-2^{nd}\) harmonic tends to 0 as \( \Omega \) approaches \( \Omega = 0.4 \). Note that rigid body motion may occur at other frequencies satisfy \( \Omega + q \beta_m = 0 \).

Frequency conversion is also observed when the incident wave and the modu-
The modulation wave have opposite direction. The reverse of the modulation wave direction makes the two stop bands of $u_0^+$ and $u_0^-$ modes exchange with each other, as can be seen from Figure 5.8(a) and (c). Inside the stop band of $u_0^-$ mode, from Figure 5.8(c) and (d), we can see that the dominant harmonic is changed from the $0^{th}$ to the $1^{st}$ after the reflection. In this case we have $\beta_m < 0$, which means frequency of the $1^{st}$ harmonic is $\Omega - |\beta_m|$, inside the stop band of $u_0^+$ mode. Therefore, in this case the main frequency is down-converted from the stop band of $u_0^-$ mode to that of $u_0^+$ mode after the reflection.

![Figure 5.8](image_url)

Figure 5.8: Case 1: harmonic amplitudes of the (a) incident and (b) reflected waves at $x_1 = -10\lambda_m$ when the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = 0.2$, propagating in the positive direction. Case 2: harmonic amplitudes of the (c) incident and (d) reflected waves at $x_1 = -10\lambda_m$ when the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = -0.2$, propagating in the negative direction. Amplitudes in both cases are normalized by the corresponding amplitude $|U_0^+ e^{-ik_0 x_1}|$ of the $0^{th}$ harmonic of the incident wave. Frequency of the $q^{th}$ harmonic is $\Omega + q\beta_m$.

The cause of the above frequency conversion is explained by further analyzing the components of the reflected waves. Without losing any generality, we choose the modulation parameters as $\alpha_m = 0.4$, $\beta_m = 0.2$. Figure 5.9(a) and (b) respectively show the components of the incident and reflected waves at frequency $\Omega_0 = 0.49$, which is the center between the two stop bands (see Figure 5.8(a) or (b)). Figure 5.9(c) and (d) respectively show those at $\Omega_1 = 0.384$, which is the center of the stop band of $u_0^-$ mode. According to Equation (5.11) and (5.12) we can see that the incident and reflected waves can be treated both as a group of Bloch modes and as a group of harmonics. These dual properties are shown in Figure 5.9. The
vertical axis indicates the orders of Bloch modes \((u_n^+ in Figure 5.9(a) and (c), u_n^- in Figure 5.9(b) and (d))\), the horizontal one represents the orders of harmonics composing corresponding modes. Therefore, the pixel \((n, q)\) represents the \(q^{th}\) harmonic of the \(n^{th}\) mode, and the color of it indicates the corresponding normalized amplitude, which is \(|U_{(0,0)}^+ e^{-i(k_0^+ + qk_m)x_1}|/|U_{(0,0)}^+ e^{-i k_0^+ x_1}| \) (see Equation (5.11)) in Figure 5.9(a) and (c), \(|B_n U_{(n,q)}^- e^{-i(k_0^- + qk_m)x_1}|/|U_{(0,0)}^+ e^{-i k_0^+ x_1}| \) (see Equation (5.12)) in Figure 5.9(b) and (d). From Figure 5.9(a) and (b) we can see that, when the frequency is outside the stop band of \(u_0^+\) mode, after the reflection, most of the energy is transmitted from the \(u_0^+\) mode to the \(u_0^-\) mode, which is dominated by the \(0^{th}\) harmonic because of frequency \(\Omega_0\). Accordingly, the main frequency is not converted in this case. However, within the stop band of \(u_0^-\) mode, the reflection makes the energy being transmitted from the \(u_0^-\) mode to the \(u_{-1}^+\) mode dominated by the \(1^{st}\) harmonic be of frequency \(\Omega_1 + \beta_m\) (Figure 5.9(c) and (d)). Therefore, the frequency conversion at ends of modulated beams is caused by energy transmission between different orders Bloch modes.

Figure 5.9: (a), (b): Components of the (a) incident and (b) reflected waves at \(\Omega_0 = 0.49\). (c), (d): Components of the (c) incident and (d) reflected waves at \(\Omega_1 = 0.384\). The \((n, q)\) pixel represents the \(q^{th}\) harmonic of the \(n^{th}\) mode composing the wave, and the color of it indicates the corresponding amplitude normalized by the amplitude \(|U_{(0,0)}^+ e^{-i k_0^+ x_1}| \) of the \(0^{th}\) harmonic of the incident wave. The modulation parameters are \(\alpha_m = 0.4, \beta_m = 0.2\).
5.3.2.2 Frequency conversion at interfaces between homogeneous and time-space modulated beams

The theory developed in Section 5.3.1.2 is used to study the frequency conversion at interfaces between homogeneous and time-space modulated beams in this section. The truncation orders are chosen as $N = 5$ and $M = 4$ to take into account all significant harmonics. Assume that a single harmonic $u_i(x,t)$ with frequency $\Omega$ is incident on the time-space modulated beam, as shown in Figure 5.7. The harmonic amplitudes $|R_q|$ of the reflected waves (see Equation (5.18)) at $x = 0$ are shown in Figure 5.10. Specifically, Figure 5.10(a) shows the results when the incident harmonic and modulation wave have the same direction ($\alpha_m = 0.4$, $\beta_m = 0.2$) and Figure 5.10(b) are the results of an opposite situation ($\alpha_m = 0.4$, $\beta_m = -0.2$). All amplitudes are normalized by the amplitude of the incident harmonic.

From Figure 5.10 we can see that, in both cases at most of the frequencies, the reflected waves are dominated by the $-1^{st}$ harmonic. When the incident harmonic and the modulation wave have the same direction (Figure 5.10(a)), frequency of the $-1^{st}$ harmonic is $\Omega - \beta_m$ ($\beta_m > 0$), which means the frequency is down converted after the reflection. On the other hand, when the incident harmonic and the modulation wave have opposite direction (Figure 5.10(b)), the $-1^{st}$ harmonic has frequency $\Omega + |\beta_m|$ ($\beta_m < 0$). Therefore, in this case the frequency is up converted.

It should be noted that even though the frequency conversion at the interface can be observed at frequencies far from the stop band of $u_0^+$ mode when the harmonic is incident on the modulated beam from the left side, the reflected and converted harmonic is a very small part of the incident one. Only in the vicinity of and within the stop band of $u_0^+$ mode, the reflection is significant, as well as the frequency conversion.

To explain the frequency conversion at the interface, the components of the induced waves inside the modulated beam are studied. We choose the modulation parameters as $\alpha_m = 0.4$, $\beta_m = 0.2$. We perform the simulations at two frequencies. The first one $\Omega_0 = 0.49$ is the center between the two stop bands in Figure 5.10(a) and the second one $\Omega_2 = 0.584$ is the center of the stop band of $u_0^+$ mode. Figure 5.11 shows the components of the reflected waves and induced waves in the modulated beam at these two frequencies. Similar to Figure 5.9, in Figure 5.11 the pixel $(n,q)$ represents the $q^{th}$ harmonic of the $n^{th}$ Bloch mode (we call the harmonics in the homogeneous beam the $0^{th}$ Bloch mode in Figure 5.11(a) and (c), modes in Figure 5.11(b) and (d) are $u_n^+$), the color of it indicates the corresponding amplitude, which is $|R_q|$ (see Equation (5.18)) in Figure 5.11(a) and (c), $|T_n U_{(n,q)}^+|$ (see Equation (5.17)) in Figure 5.11(b) and (d). These amplitudes are normalized by the amplitude of the incident harmonic. There are both positive and negative-going harmonics in the induced waves, they are respectively indicated by the signs "+" and "−" in Figure 5.11(b) and (d). In addition, some of the induced harmonics in Figure 5.11(d) (namely, components of the $u_0^+$ ($\Omega_2 = 0.584$) mode) are evanescent, they are distinguished from the propagative ones through the red and green colors of the ± signs.
5.3. Frequency conversion

Figure 5.10: Harmonic amplitudes of the reflected waves at the interface between a homogeneous beam and a time-space modulated beam. (a): the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = 0.2$, propagating in the positive direction; (b): the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = -0.2$, propagating in the negative direction. All amplitudes are normalized by the amplitude of the incident harmonic. Frequency of the $q^{th}$ harmonic is $\Omega + q\beta_m$.

Figure 5.11(b) shows that, at $\Omega_0 = 0.49$, when the harmonic is incident on the interface, most of it is transmitted into the positive-going harmonic $(0,0)$ in the modulated beam with the frequency being unaltered. Also we can see that the negative-going harmonic $(0,-1)$ be of frequency $\Omega_0 - \beta_m$ is generated. This harmonic reenters the homogeneous part, consequently leading to the observed frequency conversion. Similarly, at $\Omega_2 = 0.584$, from Figure 5.11(d) we can see that, the induced waves in the modulated beam is dominated by the harmonics $(0,0)$ and $(0,-1)$, which are all evanescent. The one $(0,0)$ is a positive-going harmonic be of frequency $\Omega_2$, it rapidly decays inside the modulated beam. On the contrary, the harmonic $(0,-1)$ is a negative-going one be of frequency $\Omega_2 - \beta_m$, amplitude of it gains toward the interface $x = 0$. The harmonic $(0,-1)$ is transmitted into the homogeneous part, causing the observed frequency conversion.

Both in Figure 5.11(b) and (d), the harmonic $(0,-1)$ could be explained as the reflected positive-going one $(0,0)$ inside the modulated beam. This reflection is caused by the Bragg scattering effect, which must occur at any section of the modulated beam due to the continuous periodic variation of the impedance introduced by the modulation of Young’s modulus. Therefore, the frequency conversion observed at the interface is due to the Bragg scattering effect inside the time-space modulated beam.

5.3.3 Numerical simulations

The above theoretically studied frequency conversion phenomena are numerically verified by using the finite element method in this section. In all the numerical
Figure 5.11: (a), (b): Components of the (a) reflected waves and (b) induced waves in the modulated beam at $\Omega_0 = 0.49$. (c), (d): Components of the (c) reflected waves and (d) induced waves in the modulated beam at $\Omega_2 = 0.584$. The $(n, q)$ pixel represents the $q^{th}$ harmonic of the $n^{th}$ mode composing the wave, and the color of it indicates the corresponding amplitude normalized by the amplitude of the incident harmonic. + and - signs indicate positive and negative-going harmonics, respectively, the red and green colors of them distinguish evanescent and propagative harmonics. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$.

studies, each $\lambda_m$ length is discretized by 20 2D Lagrange elements. The Generalized $\alpha$ method is used to evaluate the time-domain response, a fixed time step equal to $0.0025\lambda_m/c_0$ is used.

To verify the frequency conversion at the ends, the $2L$ ($L = 100\lambda_m$) long time-space modulated beam shown by the top figure in Figure 5.12 is considered. Both the two ends of the beam are free. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. A narrow band tone burst load centered at $\Omega_1 = 0.384$ (inside the stop band of $u_{-0}$ mode) is applied at the left end along the $x$ direction to generate longitudinal waves. The rest three panels show the spectra of waves in the beam at 3 successive instants, arrows indicate the wave propagation directions. It can be seen that the main frequency of the generated waves is $\Omega_1 = 0.384$ ($t = 100\lambda_m/c_0$). When these waves are reflected by the right end, the main frequency is up converted to $\Omega_2 = 0.584$, which is inside the stop band of $u_{+0}$ mode ($t = 305\lambda_m/c_0$ and $355\lambda_m/c_0$). The frequency difference caused by the conversion is exactly equal to the modulation frequency, which is $\Omega_m = \beta_m = 0.2$, as predicted by the theoretical studies.

The frequency conversion at the interface is verified in Figure 5.13. The top figure shows the considered model with two ends both free. The left part of the
Figure 5.12: Frequency conversion at ends of time-space modulated beams. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. The length of the beam is $2L$, $L = 100\lambda_m$. Both the two ends of the beam are free. Arrows indicate wave propagation directions.

beam ($-L \leq x < 0$) has uniform materials, while the right part ($0 \leq x \leq L$) is a time-space modulated beam with $\alpha_m = 0.4$, $\beta_m = 0.2$. A narrow band tone burst load centered at $\Omega_2 = 0.584$ is applied at the left end to generate longitudinal waves ($t = 50\lambda_m/c_0$). At the interface ($x = 0$), most of these waves are reflected back with down converted frequency $\Omega_1 = 0.384$ ($t = 150\lambda_m/c_0$). These reflected waves propagate toward the left end and are then reflected back by this static end with unchanged amplitudes and frequencies. Since the main frequency of these waves are now outside the stop band of $u_0^+$ mode, most of them are transmitted into the modulated beam ($t = 400\lambda_m/c_0$).

5.4 Conclusions

This chapter studied the wave propagation in time-space modulated structures. First, the dispersion relations and Bloch modes were studied. The symmetry of time-space modulated structures’ band diagrams is broken, i.e., it is impossible to find a wave that can propagate in two opposite directions in modulated structures. Also due to the symmetry-broken, the stop bands of different orders Bloch modes occupy different frequency ranges. The relative distance with respect to the frequency between two adjacent stop bands depends on the modulation velocity.

Bloch modes in time-space modulated structures can be expressed as groups of harmonics be of different frequencies and wave numbers. At frequencies far from the stop band, the modes are dominated by one harmonic. In the vicinity of and within the band, the modes are dominated by two harmonics with opposite directions. The
Chapter 5. Wave propagation in time-space modulated structures

Figure 5.13: Frequency conversion at interfaces between homogeneous and time-space modulated beams. The left part of the beam ($-L \leq x < 0$) has uniform materials with $\alpha_m = \beta_m = 0$, the right part ($0 \leq x \leq L$) is a time-space modulated structure with $\alpha_m = 0.4$, $\beta_m = 0.2$, $L = 100\lambda_m$. Both the two ends of the beam are free. Arrows indicate wave propagation directions.

harmonics of modes in time-space modulated structures also present some peculiar properties compared with those in space-only periodic structures. At 0 Hz, the time-space modulation gives the harmonics except the $0^{th}$ one an initial velocity equal to the modulation velocity. As a consequence, some of these harmonics’ directions inverse as the frequency increases.

This chapter also demonstrated and explained two types of frequency conversion induced by time-space modulated structures. The first type is caused by energy transmission between different orders Bloch modes, it can be observed when interior waves are reflected by boundaries of time-space modulated structures. The second type is due to the Bragg scattering effect inside the modulated structures, it can be observed when external waves are reflected by time-space modulated structures.

The frequency can be up or down converted, and the frequency difference is equal to the modulation frequency. In the first type of conversion, when the incident and modulation waves are co-propagating, frequency up-conversion is observed after the reflection. On the other hand, frequency down-conversion is observed when the incident and modulation waves are count-propagating. In the second type, the frequency conversion direction is totally reversed. The co-propagating incident and modulation waves lead to frequency down-conversion, and the count-propagating incident and modulation waves yield frequency up-conversion.

The frequency conversion has significant influences on practical applications of time-space modulated structures. It may need to be taken into account in applications using the strong non-reciprocity reported in [93, 163]. For example, in
5.4. Conclusions

approximate infinite or semi-infinite systems, this strong non-reciprocity may could be exploited to build unidirectional insulators. However, when the harmonics scattered by the modulated structures are considerably reflected back, the one-way energy insulation will fail due to the frequency conversion, as implied in Figure 5.13. Nevertheless, the frequency difference caused by the conversion depends on the modulation frequency (or say the modulation speed), which is tunable. Therefore, the frequency conversion could be exploited to manipulate frequencies of waves for particular purposes.
Chapter 6

Reflection and transmission of elastic waves incident on time-space modulated structures

Contents

6.1 Bloch modes in modulated beams .......................... 120
6.2 Reflection and transmission of elastic waves incident on modulated beams ........................................ 121
   6.2.1 Introduction to the scattering matrix method ........ 121
   6.2.2 Extending the scattering matrix method to modulated beams 121
   6.2.3 Energy flux ........................................ 124
6.3 Results .................................................. 125
   6.3.1 Validation of the extended scattering matrix method 125
   6.3.2 Frequency reflection and transmission properties ..... 127
   6.3.3 Insights into the unusual phenomena induced by modulated beams ........................................ 129
   6.3.4 Influences of the modulation velocity .................. 132
   6.3.5 Energy balance analysis ................................ 134
   6.3.6 The feasibility of one-way energy insulation in finite systems 139
6.4 Conclusions ............................................. 142

Building one-way energy insulators (mechanical diodes) perhaps is the most attractive application of time-space modulated structures. In this chapter, reflection and transmission of elastic waves incident on time-space modulated structures of finite length are studied to fully evaluate this application. The longitudinal motion in slender beams is considered. Note that our studies can be easily extended to other types of guided elastic waves, like flexural waves in beams. In what follows, time-space modulated structures will be simply called modulated structures. In Section 6.1, the expressions of the Bloch modes in modulated beam are recalled. In sections 6.2 and 6.3, reflection and transmission of longitudinal elastic waves incident on modulated beams are studied. Particularly, a theoretical method is developed (Section 6.2.2) and verified (Section 6.3.1). The properties of reflection and transmission are studied in Section 6.3.2. Insights into the unusual phenomena caused by modulated beams are given in Section 6.3.3. Section 6.3.4 discusses the influences
of the modulation velocity. Section 6.3.5 is dedicated to the energy balance of systems containing modulated structures. The feasibility of using modulated structures to realize one-way energy insulation in finite systems is discussed in Section 6.3.6. Finally, important conclusions of this chapter are summarized in section 6.4.

6.1 Bloch modes in modulated beams

The time-space modulated beam introduced in Section 5.1 of Chapter 5 is used in this chapter. The density of the beam $\rho_0$ is constant and homogeneous, while the Young’s modulus is modulated in time and space according to a cosine wave function:

$$E(x,t) = E_0 + E_m \cos(\omega_m t - k_m x)$$  \hspace{1cm} (6.1)

Recall that, $E_0$ is the Young’s modulus when there is no modulation, $E_m$ is the modulation amplitude, $\omega_m$ and $k_m$ are respectively the angular frequency and wavenumber of the modulation wave, whose wavelength is $\lambda_m = 2\pi/k_m$. The modulation wave propagates along the beam with the speed $v_m = \omega_m/k_m$. In the following discussion, the modulation expressed in Equation (6.1) is described by two dimensionless parameters, namely the dimensionless modulation amplitude $\alpha_m = E_m/E_0$ and dimensionless modulation speed $\beta_m = v_m/c_0$, here $c_0 = \sqrt{E_0/\rho_0}$.

As introduced in Section 5.2, the longitudinal Bloch modes in the modulated beam can be obtained by using the Plane Wave Expansion (PWE) method, which leads to a Quadratic Eigenvalue Problem (QEP): $Q(\omega, k)\tilde{U} = 0$. When the truncation order is $N$, $4N + 2$ longitudinal Bloch modes are solved from the QEP. These Bloch modes are classified into positive-going and negative-going groups. The $n^{th}$ Bloch modes in the positive-going and negative-going groups are respectively represented by $u_n^+$ and $u_n^-$, with $n = -N, ..., 0, ..., +N$. They are expressed as groups of harmonics:

$$u_n^+(x, t, k_n^+, \omega) = \sum_{q=-N}^{+N} U_{(n,q)}^+ e^{i[(\omega+q\omega_m)t-(k_n^+ + qk_m)x]}$$

$$u_n^-(x, t, k_n^-, \omega) = \sum_{q=-N}^{+N} U_{(n,q)}^- e^{i[(\omega+q\omega_m)t-(k_n^- + qk_m)x]}$$  \hspace{1cm} (6.2)

in which, $k_n^\pm$ are eigenvalues obtained from the QEP, $\{U_{(n,-N)}^+, ..., U_{(n,0)}^+, ..., U_{(n,+N)}^+\}^T$ and $\{U_{(n,-N)}^-, ..., U_{(n,0)}^-, ..., U_{(n,+N)}^-\}^T$ are the corresponding eigenvectors.
6.2 Reflection and transmission of elastic waves incident on modulated beams

6.2.1 Introduction to the scattering matrix method

Figure 6.1 illustrates the scattering of incident waves by an 1D scatterer. The scatterer is between the waveguide 1 and 2. \( u_0^+ \) and \( u_0^- \) indicate the incident waves from these two waveguides, respectively. \( u_s^- \) and \( u_s^+ \) indicate the scattered waves. These incident and scattered waves are related by a scattering matrix in the following manner:

\[
\begin{bmatrix}
    u_s^- \\
    u_s^+
\end{bmatrix}
= \begin{bmatrix}
    R_{11} & T_{21} \\
    T_{12} & R_{22}
\end{bmatrix}
\begin{bmatrix}
    u_0^+ \\
    u_0^-
\end{bmatrix}
\]  
(6.3)

in which, \( R_{11}, T_{12}, T_{21} \) and \( R_{22} \) are frequency-dependent coefficients [77].

The scattering relations in Equation (6.3) can be used to obtain the scattered waves. Also the scattering matrix is very useful to describe the reflection and transmission properties of the scatterer. \( R_{11} \) and \( T_{12} \) are the reflection and transmission coefficients corresponding to the incident wave from the waveguide 1. On the other hand, \( T_{21} \) and \( R_{22} \) are the transmission and reflection coefficients corresponding to the incident wave from the waveguide 2. Using these features, the scattering matrix is proposed to judge whether a so-called non-reciprocal device is really non-reciprocal [164, 165].

6.2.2 Extending the scattering matrix method to modulated beams

In this section, the scattering matrix method is extended to study the reflection and transmission of elastic waves incident on time-space modulated structures. The finite modulated beam in Figure 6.2 occupying the region \( x_1 < x < x_2 \) is considered, the left and right ends of it are respectively connected to a semi-infinite uniform beam. The Young’s modulus and density of the modulated beam are the same as in Section 6.1, those of the uniform beams are \( E_0 \) and \( \rho_0 \).

Assume that two longitudinal harmonics \( u_0^+ \) and \( u_0^- \) are respectively incident on the modulated beam from left and right sides as shown in Figure 6.2:

\[
\begin{align*}
    u_0^+(x,t) &= A_0 e^{i\omega(t-x/c_0)} \\
    u_0^-(x,t) &= G_0 e^{i\omega(t+x/c_0)}
\end{align*}
\]  
(6.4)
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

Figure 6.2: Scattering of incident waves by a modulated beam.

here, $A_0$ and $G_0$ are corresponding amplitudes, which are known for particular simulations.

Positive-going ($u_m^+$) and negative-going ($u_m^-$) waves are induced inside the modulated beam. Mathematically these induced waves can be represented as superposition of the Bloch modes supported by the beam. When $2R + 1$ Bloch modes are used, according to Equations (6.2), these waves are represented as:

$$u_m^+(x,t) = \sum_{n=-R}^{+R} C_n u_n^+ e^{i[(\omega+q\omega_m)t-(k_n^+q)k+q)k]x}$$

$$u_m^-(x,t) = \sum_{n=-R}^{+R} D_n u_n^- e^{i[(\omega+q\omega_m)t-(k_n^-q)k+q)k]x}$$

(6.5)

in which, $C_n$ and $D_n$ are contribution coefficients of corresponding Bloch modes. Note that, the number of Bloch modes used in Equations (6.5) must satisfy $R \leq N - 1$ (see Section 5.3.1.2).

The scattered waves ($u_s^-$ and $u_s^+$) are superposition of harmonics with all possible frequencies $\omega + q\omega_m$:

$$u_s^-(x,t) = \sum_{q=-R}^{+R} B_q e^{i(\omega+q\omega_m)(t+\frac{x}{c_0})}$$

$$u_s^+(x,t) = \sum_{q=-R}^{+R} F_q e^{i(\omega+q\omega_m)(t-\frac{x}{c_0})}$$

(6.6)

$B_q$ and $F_q$ are amplitudes of corresponding harmonics.

The incident waves ($u_i^+$, $u_i^-$) and scattered waves ($u_s^-$, $u_s^+$), as well as the induced waves ($u_m^+$, $u_m^-$) inside the modulated beam must satisfy the continuity of displacement and of force at interfaces $x_1$, $x_2$. These conditions are:
6.2. Reflection and transmission of elastic waves incident on modulated beams

\[ u_i^+(x_1, t) + u_s^-(x_1, t) = u_m^+(x_1, t) + u_m^-(x_1, t) \]

\[ E_0 \frac{\partial [u_i^+(x, t) + u_s^-(x, t)]}{\partial x} \bigg|_{x=x_1} = E(x_1, t) \frac{\partial [u_m^+(x, t) + u_m^-(x, t)]}{\partial x} \bigg|_{x=x_1} \]

\[ u_m^+(x_2, t) + u_m^-(x_2, t) = u_s^+(x_2, t) + u_s^-(x_2, t) \]

\[ E(x_2, t) \frac{\partial [u_m^+(x, t) + u_m^-(x, t)]}{\partial x} \bigg|_{x=x_2} = E_0 \frac{\partial [u_s^+(x, t) + u_s^-(x, t)]}{\partial x} \bigg|_{x=x_2} \]

Substituting the expressions of waves (Equations (6.4) to (6.6)) and the 2D Fourier expansion of the Young's modulus of the modulated beam (Equation (6.1)) into the continuity conditions results in:

\[ A_0 e^{-i \frac{\omega x_1}{c_0}} e^{i \omega t} + \sum_{q=-R}^{+R} B_q e^{-i \frac{q \omega m x_1}{c_0}} \]

\[ - \sum_{n=-R}^{+R} \left[ C_n U_{(n,q)}^+ e^{-i(k_n^+ + q k_m) x_1} + D_n U_{(n,q)}^- e^{-i(k_n^- + q k_m) x_1} \right] e^{i(\omega + q \omega_m) t} = 0 \]

\[ - \frac{\omega}{c_0} A_0 e^{-i \frac{\omega x_1}{c_0}} e^{i \omega t} + \sum_{q=-R}^{+R} \left\{ \frac{\omega}{c_0} + \frac{q \omega m}{c_0} B_q e^{-i \frac{q \omega m x_1}{c_0}} \right\} \]

\[ + \sum_{n=-R}^{+R} \sum_{p=-1}^{+1} \hat{E}_p e^{-i p k_m x_1} \left[ C_n (k_n^+ + (q - p) k_m) U_{(n,q-p)}^+ e^{-i(k_n^+ + (q-p) k_m) x_1} \right. \]

\[ + D_n (k_n^- + (q - p) k_m) U_{(n,q-p)}^- e^{-i(k_n^- + (q-p) k_m) x_1} \] \left. \right] e^{i(\omega + q \omega_m) t} = 0 \]

\[ \left\{ \sum_{q=-R}^{+R} \left\{ \sum_{n=-R}^{+R} \left[ C_n U_{(n,q)}^+ e^{-i(k_n^+ + q k_m) x_2} + D_n U_{(n,q)}^- e^{-i(k_n^- + q k_m) x_2} \right] \right. \]

\[ - F_q e^{-i \frac{\omega q m x_2}{c_0}} \} - G_0 e^{i \frac{\omega q m x_2}{c_0}} \} e^{i(\omega + q \omega_m) t} = 0 \]

\[ \left\{ \sum_{q=-R}^{+R} \left\{ \sum_{n=-R}^{+R} \left\{ \sum_{p=-1}^{+1} \hat{E}_p e^{-i p k_m x_2} \left[ C_n (k_n^+ + (q - p) k_m) U_{(n,q-p)}^+ e^{-i(k_n^+ + (q-p) k_m) x_2} \right. \right. \right. \]

\[ + D_n (k_n^- + (q - p) k_m) U_{(n,q-p)}^- e^{-i(k_n^- + (q-p) k_m) x_2} \right. \]

\[ - E_0 \frac{\omega}{c_0} F_q e^{-i \frac{\omega q m x_2}{c_0}} \} + E_0 \frac{\omega}{c_0} G_0 e^{i \frac{\omega q m x_2}{c_0}} \} e^{i(\omega + q \omega_m) t} = 0 \]

in which, \( \hat{E}_p \) (\( p = -1, 0, 1 \)) are the Fourier coefficients in the Fourier expansion of the modulated Young’s modulus.
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

By exploiting the orthogonality of harmonic functions \( e^{i(\omega + q\omega_m)t} \), the four equations in (6.8) can be rewritten into the following matrix forms:

\[
\begin{align*}
M_{B_1}B - M_{C_1}C - M_{D_1}D &= -A_0M_{A_1}I_0 \\
M_{B_2}B + M_{C_2}C + M_{D_2}D &= A_0M_{A_2}I_0 \\
M_{C_3}C + M_{D_3}D - M_{F_3}F &= G_0M_{G_3}I_0 \\
M_{C_4}C + M_{D_4}D - M_{F_4}F &= -G_0M_{G_4}I_0
\end{align*}
\] (6.9)

where, \( B, C, D \) and \( F \) are column vectors respectively containing coefficients \( B_q, C_n, D_n \) and \( F_q, n, q = -R, ..., 0, ..., R \). \( I_0 \) is a \((2R + 1)\)-by-1 vector, with \( I_0(R + 1) = 1 \) and other elements are zero. Dimensions of matrices represented by \( M_i \) are all \((2R + 1)\)-by-\((2R + 1)\). Details of them are listed in Appendix E.

By solving Equations (6.9), one can obtain the coefficients vectors \( B, C, D \) and \( F \), then using the expressions in Equations (6.5), (6.6), one can obtain the scattered waves, as well as the induced waves inside the modulated part.

To further establish the scattering relations between the incident and scattered waves, coefficients \( C \) and \( D \) in Equations (6.9) are eliminated and the rest equations are rewritten as:

\[
\begin{bmatrix} B \\ F \end{bmatrix} = \begin{bmatrix} H_{BF11} & H_{BF12} \\ H_{BF21} & H_{BF22} \end{bmatrix}^{-1} \begin{bmatrix} H_{AG11} & H_{AG12} \\ H_{AG21} & H_{AG22} \end{bmatrix} \begin{bmatrix} A_0 \\ G_0 \end{bmatrix}
\] (6.10)

Details of the matrices in above equations are in Appendix E.

Equations (6.10) are the final scattering relations. The corresponding scattering matrix is:

\[
S = \begin{bmatrix} H_{BF11} & H_{BF12} \\ H_{BF21} & H_{BF22} \end{bmatrix}^{-1} \begin{bmatrix} H_{AG11} & H_{AG12} \\ H_{AG21} & H_{AG22} \end{bmatrix}
\] (6.11)

The scattering matrix \( S \) is a \((4R+2)\)-by-2 matrix, as will be shown in the Section 6.3, it is a very handy tool to characterize the reflection and transmission properties of the modulated structures.

6.2.3 Energy flux

Assume that there is only one harmonic incident from the left side, namely, \( A_0 \neq 0 \) and \( G_0 = 0 \). The reflected and transmitted waves are respectively \( u_-(x,t) \) and \( u_+(x,t) \) in Equation (6.6).

The energy flux at the interface \( x_1 \) caused by the incident harmonic is:

\[
I_i(t) = E_0 \frac{\partial u_+(x,t)}{\partial x} \bigg|_{x=x_1} \cdot \frac{\partial u_+(x,t)}{\partial t} = \frac{E_0\omega^2 A_0^2}{c_0} e^{2\omega(t-\frac{x}{C_0})}
\] (6.12)
The energy flux of the reflected waves at interface \( x_1 \) is:

\[
I_r(t) = E_0 \frac{\partial u_-(x,t)}{\partial x} \bigg|_{x=x_1} \cdot \frac{\partial u_-(x,t)}{\partial t} \\
= -E_0 \sum_{q_1=-R}^{+R} \sum_{q_2=-R}^{+R} B_{q_1} B_{q_2} (\omega + q_1 \omega_m) (\omega + q_2 \omega_m) c_0 2i(\omega + q_1 \omega_m)(t - \frac{q_1}{c_0})
\]

(6.13)

The energy flux of the transmitted waves at interface \( x_2 \) is:

\[
I_t(t) = E_0 \frac{\partial u_+^+(x,t)}{\partial x} \bigg|_{x=x_2} \cdot \frac{\partial u_+^+(x,t)}{\partial t} \\
= E_0 \sum_{q_1=-R}^{+R} \sum_{q_2=-R}^{+R} F_{q_1} F_{q_2} (\omega + q_1 \omega_m) (\omega + q_2 \omega_m) c_0 2i(\omega + q_1 \omega_m)(t - \frac{q_2}{c_0})
\]

(6.14)

Similarly, we can obtain the incident, reflected and transmitted energy when a harmonic is incident from the right side.

### 6.3 Results

#### 6.3.1 Validation of the extended scattering matrix method

The Finite Element (FE) method is used as benchmark to verify the extended scattering matrix method developed in Section 6.2.2. Figure 6.3 shows the beam model used in the FE simulations. It is discretized by 2D Lagrange elements, there are 20 elements per \( \lambda_m \) length. The left and right parts of the model are uniform (homogeneous) beams, the central part is a modulated beam. Absorption Boundary Conditions (ABC) are used to avoid reflection at the beam ends, therefore to mimic the scattering problem illustrated in Figure 6.2. Since there is a time-varying material parameter in the model, the traditional frequency-domain analysis by performing Fourier transform on the whole governing equations is not available. Therefore, in the FE simulations, the time-domain analysis is used. The responses are evaluated by using the generalized \( \alpha \) method with a fixed time step equal to 0.0025\( \lambda_m/c_0 \). In the theoretical simulations using the extended scattering matrix method, the incident wave is a pure harmonic. Therefore, to obtain an approximate pure incident harmonic in the FE simulations, tone-burst excitations are used, and the duration of the excitations is long to centralize the spectra of the generated waves.

In the FE simulations, modulation parameters of the modulated beam are chosen as \( \alpha_m = 0.4, \beta_m = 0.2 \). The lengths of the uniform and modulated parts in Figure 6.3 are respectively \( L_u = 10\lambda_m \) and \( L_m = 20\lambda_m \). The tone-burst excitation \( F(t) \) (with the spectrum centred at \( \Omega_e \)) is applied at the left end of the model to generate longitudinal waves incident from the left side of the modulated beam. The responses of the model in time domain are evaluated; then the spectra at two observation points, namely \( O_L \) and \( O_R \) in Figure 6.3, are obtained by performing FFT on the passing waves at these two points. Note that waves at point \( O_L \) are superposition of the incident and reflected waves, those at point \( O_R \) are the transmitted waves.
Chapter 6. Reflection and transmission of elastic waves incident on
time-space modulated structures

Figure 6.3: The beam model used in the FE simulations. The left and right parts are uniform beams, the central part is a modulated beam. Absorption Boundary Conditions (ABC) are applied at the two ends. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The lengths are $L_u = 10\lambda_m$ and $L_m = 20\lambda_m$. $O_L$ and $O_R$ are the two observation points, which are the centers of the two uniform beams, respectively.

These spectra are compared with the theoretical results, which are obtained by solving Equations (6.10) with $G_0 = 0$ at $\Omega_e$. (One can also solve Equations (6.9) to have these results). In the theoretical simulations, the modulation parameters and length of the modulated beam are the same to those in the FE simulations. In addition, the truncation order used in the theoretical method is $R = 4$. It will be seen that this truncation order is enough to take into account all the significant scattered harmonics. Thus, it is used in all the following theoretical simulations.

Figures 6.4 and 6.5 show the comparison between the numerical and theoretical results at a frequency inside the first stop band of $u^+_0$ mode ($\Omega_e = 0.584$) and at a frequency outside all the stop bands ($\Omega_e = 0.49$), respectively. The first frequency is the center of the stop band of $u^+_0$ mode, and the second one is the center between the stop bands of $u^+_0$ and $u^-_0$ modes (see Figure 5.4). In Figures 6.4 and 6.5, the black lines represent the results obtained by the FE method. The red matches are the results from the theoretical method. The length of them indicates the amplitude value. To facilitate comparison, all the results are normalized by the maximum amplitude. $\Omega_m = \beta_m$ is the dimensionless modulation frequency.

It can be seen that the extended scattering matrix method can accurately predict the reflected and transmitted harmonics. Figures 6.4(a) and 6.4(b) show discrepancies at some crests. These disagreements are caused by the inconsistency between the incident waves in the theoretical and FE simulations. In particular, as can be seen in Figures 6.4(a) and 6.4(b), when the excitation frequency is $\Omega_e = 0.584$, most of the incident waves are reflected by the modulated beam. (More details about the reflection and transmission will be discussed in the following section). These reflected waves will interact with the source in the FE simulation (for example, parts of these waves will be reflected at the source point due to the discontinuity and make contribution to the incident waves), consequently causing the differences between the incident waves in the theoretical and FE simulations. This interpretation is supported by the better agreement shown in Figures 6.5(a) and 6.5(b), in which case, the reflection from the modulated beam is weak hence the incident harmonic is less disturbed in the FE simulation.
6.3. Results

(a) spectra at point $O_L$

(b) spectra at point $O_R$

Figure 6.4: Comparison between the results obtained by the FE method and the theoretical method when the modulated beam is stimulated by a left incident harmonic at $\Omega_e = 0.584$. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The dimensionless modulation frequency is $\Omega_m = 0.2$. The length of the modulated beam is $L_m = 20\lambda_m$.

6.3.2 Frequency reflection and transmission properties

In this section, the frequency reflection and transmission properties within a frequency range are studied. The scattering matrix in equation (6.11) can be divided into four equal quadrants:

$$S = \begin{bmatrix} R_L & T_R \\ T_L & R_R \end{bmatrix} \quad (6.15)$$

elements in quadrants $R_L$ and $T_L$ are respectively Reflection and Transmission coefficients of a harmonic incident from the Left side. Elements in quadrants $T_R$ and $R_R$ are respectively Transmission and Reflection coefficients of a harmonic incident from the Right side.

Figures 6.6 and 6.7 show the reflection and transmission coefficients at different frequencies corresponding to a short modulated beam ($L_m = 20\lambda_m$) and a long
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

Figure 6.5: Comparison between the results obtained by the FE method and the theoretical method when the modulated beam is stimulated by a left incident harmonic at $\Omega_c = 0.49$. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The dimensionless modulation frequency is $\Omega_m = 0.2$. The length of the modulated beam is $L_m = 20\lambda_m$.

modulated beam ($L_m = 40\lambda_m$), respectively. In these figures, each curve is associated with a harmonic composing the reflected or transmitted waves of frequency $\Omega + q\Omega_m$.

From Figure 6.6 we can see that, when a harmonic inside the stop band of $u_{0}^{+}$ mode is incident from the left side, most of it will be reflected (see Figure 6.6(a)). In contrary, when this harmonic is incident from the right side, most of it will be transmitted (see Figure 6.6(b)). This non-reciprocal transmission is also observed at frequency inside the stop band of $u_{0}^{-}$ mode. In this case, a harmonic from the right side will be mostly reflected, but the same harmonic from the left side will be mostly transmitted.

Also, some nonlinear phenomena occur during the reflection and transmission. It can be seen that, no matter which direction the harmonic is incident from, the reflected and transmitted waves both contain multiple harmonics with frequencies $\Omega + q\Omega_m$. This frequency splitting phenomenon is more obvious inside the two stop
bands. Particularly, high order harmonics are observed in the transmitted waves when strong reflection occurs. Increasing the length of the modulated beam can’t weaken these harmonics, which means that this transmission is inevitable in modulated structures. The second nonlinear phenomenon is the frequency conversion observed in the reflected waves. In particular, the left incident harmonic is mainly reflected into the $-1^{st}$ harmonic be of frequency $\Omega - \Omega_m$ (see the left figure in Figure 6.6(a)). On the contrary, the right incident harmonics is mainly reflected into a harmonic be of frequency $\Omega + \Omega_m$ (see the left figure in Figure 6.6(b)). The frequency conversion is especially significant in the vicinity of and within the corresponding stop bands since the reflection is more intense at these frequencies. Recall that the peak of the coefficients of the $-2^{nd}$ harmonic at $\Omega = 0.4$ is caused by the rigid body motion of the beam, since frequency $(\Omega - 2\beta_m)$ of the $-2^{nd}$ harmonic tends to 0 as $\Omega$ approaches $\Omega = 0.4$.

The strong non-reciprocal wave transmission revealed in figures 6.6 and 6.7 indicates that, modulated structures can serve as one-way energy insulators in approximate infinite systems or semi-infinite systems. For example, modulated structures could be used to protect coasts and harbors from see waves. They will let the offshore disturbance propagates to the ocean however block the impact from the ocean. The ocean can be treated as semi-infinite consequently the reflected waves will not be reflected back. Modulated structures could also be used in finite structures which can be approximated by equivalent infinite ones [166], in these structures the scattered waves are attenuated by damping or radiation therefore are not reflected back to the modulated structures.

6.3.3 Insights into the unusual phenomena induced by modulated beams

The mechanisms behind the unusual phenomena revealed in the above section are explained by further studying the induced waves inside the modulated beam. Modulation parameters are chosen as $\alpha_m = 0.4$, $\beta_m = 0.2$ and the length of the modulated beam is $20\lambda_m$. With these parameters, two simulations are done to explain the non-reciprocal transmission, frequency splitting and frequency conversion revealed in Figure 6.6. In the first simulation, a harmonic with amplitude $A_0$ is incident on the modulated beam from the left side at $\Omega_e = 0.584$ (within the stop band of $u_0^+$ mode). The coefficients $C_n$ and $D_n$ of the induced waves inside the modulated part (see Equations (6.5)) are obtained by solving Equations (6.9) with $G_0 = 0$. In the second simulation, a harmonic with amplitude $G_0$ is incident from the right side also at $\Omega_e = 0.584$. The coefficients $C_n$ and $D_n$ are obtained by solving Equations (6.9) with $A_0 = 0$.

The stimulated waves inside the modulated beam can be treated as superposition of Bloch modes, also they can be seen as groups of harmonics. The orders of Bloch modes are indicated by different indexes $n$, and the harmonic components of a Bloch mode are distinguished by different indexes $q$. In the following discussions, the index $(n, q)$ is used to indicate the $q^{th}$ harmonic component of the $n^{th}$ Bloch
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

For a harmonic incident from the left side.

For a harmonic incident from the right side.

Figure 6.6: Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$.

According to Equations (6.5), amplitudes of the $(n, q)$ harmonics inside the wave groups $u_n^+$ and $u_n^-$ are $|C_n U_{(n,q)}^+|$ and $|D_n U_{(n,q)}^-|$, respectively.

Figure 6.8 shows the induced wave groups inside the modulated beam when it is stimulated by a left and a right incident harmonic at $\Omega_e = 0.584$, respectively. The vertical axis indicates the orders of the stimulated modes, and the horizontal axis represents the orders of harmonics composing corresponding modes. The pixel $(n, q)$ in each figure represents the $(n, q)$ harmonic of the corresponding wave group, and its color indicates the amplitude. Therefore, superposition of harmonics in the $n^{th}$ row composes the $n^{th}$ Bloch mode, which is $u_n^+$ in Figures 6.8(a), 6.8(c), and $u_n^-$ in Figures 6.8(b), 6.8(d); further superposition of all the rows in each figure constitutes the induced wave group $u_m^+$ or $u_m^-$. There are positive-going and negative-going ones among the stimulated harmonics, as indicated by the signs "+" and "-" in Figure 6.8. (Only relatively significant components are marked). In addition, some of these harmonics (namely components of the $u_0^+(\Omega_e = 0.584)$ mode) are evanescent, they are distinguished from the propagative ones through the red and green colors of the ± signs.

The mechanisms behind the unusual phenomena induced by the modulated beam can be explained by analyzing the Bloch modes or harmonics shown in Figure 6.8. It can be seen that when the same harmonic is incident on the modulated beam
6.3. Results

(a) For a harmonic incident from the left side.

(b) For a harmonic incident from the right side.

Figure 6.7: Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is \( L_m = 40\lambda_m \). The modulation parameters are \( \alpha_m = 0.4, \beta_m = 0.2 \).

from opposite directions, the stimulated modes are different, not only in amplitudes but also in orders. Particularly, when the harmonic is incident from the left side, the induced wave group \( u_m^+ \) is dominated by the \( u_0^+(\Omega_e = 0.584) \) mode, which is evanescent. This mode is dominated by harmonics \((0,0)\) and \((0,-1)\) (see Figure 6.8(a)). The first one is a positive-going harmonic, it rapidly decays inside the modulated beam. The second one is a negative-going increasing harmonic of frequency \( \Omega_e - \Omega_m \), its amplitude gains toward the interface \( x_1 \). This harmonic is transmitted into the left uniform beam, leading to the large reflection of the incident harmonic and frequency down conversion. The left incident harmonic not only induces the 0th mode in the modulated beam but also generates higher order modes. These modes propagate through the modulated beam, consequently causing the inevitable transmission discussed in Section 6.3.2. In contrary, when the harmonic is incident from the right side, the induced wave group \( u_m^- \) is composed of propagative modes dominated by harmonic \((0,0)\) (see Figure 6.8(d)). Most of this wave group is transmitted through the interface \( x_1 \) into the left uniform beam (Figure 6.8(c) indicates small reflection at the interface), leading to significant transmission of the incident harmonic in this case. In both cases, it is obvious that different orders harmonics are induced inside the modulated beam. These harmonics are transmitted into the uniform beams, therefore leading to frequency splitting observed in the
Chapter 6. Reflection and transmission of elastic waves incident on
time-space modulated structures

6.3.4 Influences of the modulation velocity

The modulation is characterized by the modulation amplitude $\alpha_m$ and modulation
velocity $\beta_m$. Increasing the modulation amplitude, the width of the stop bands
will also augment [93]. Increasing the modulation velocity will further separate the
stop bands, as shown in Section 5.2.2. Therefore, tuning these parameters, we can
tune the locations of stop bands to cover desired frequency ranges, consequently
to realize one-way wave transmission at those frequencies. However, we found that,
as the modulation velocity increases, some new phenomena occur, which have sig-
nificant influences on the one-way energy transmission. These results are discussed
hereinafter.

Figures 6.9 and 6.10 respectively show the reflection and transmission coefficients
for $\beta_m = 0.4$ and $\beta_m = 0.6$. The length of the modulated beam is $20\lambda_m$ in all these

reflected and transmitted waves.
6.3. Results

Comparing the results in Figures 6.6, 6.9 and 6.10, first of all, we can see that, as the modulation velocity increases, higher order stop bands of the $u_0^-$ mode are tuned to be at low frequency range. Also those of the $u_0^+$ mode are observed at higher frequency range, which are not shown in those figures. One-way transmission can also be achieved within these bands. However, to efficiently block waves from one direction within these bands, the length of the modulated beam need to be very long. Besides, the widths of them are really not comparable to that of the first stop band. Therefore, our attention is focused on the first stop band, within which, $20\lambda_m$ is long enough to realize good one-way wave transmission.

![Figure 6.9: Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.4$.](image)

From these results, we can also see that when the harmonic is incident from the left side along the direction of the modulation wave, within the first stop band of $u_0^-$ mode, amplitudes of some harmonics of the transmitted waves increase significantly as the modulation velocity gains. In detail, amplitudes of the $-2^{nd}$, $-1^{st}$ and $1^{st}$ harmonics increase when the modulation velocity is changed from 0.2 to 0.4. Further increase of the modulation velocity leads to larger amplitudes of the above mentioned harmonics as well as those of the $0^{th}$ and $2^{nd}$ harmonics. The increase of the amplitudes of transmitted waves implies that more energy will be transmitted, i.e., the one-way energy insulation for the harmonics incident along the direction of the modulation wave becomes less efficient as the modulation velocity increases. Note that, this phenomenon is not observed within the first stop band of $u_0^+$ mode when the harmonic is incident from the right side against the modulation wave.

Reflection of waves incident on time-space modulated media was previously studied using a perturbation method by J. Simon [167]. Only the fundamental and first order harmonics were remained in his studies. He predicted that when a harmonic of frequency $\Omega$ is incident on the modulated medium, if the direction of the inci-
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

6.3.5 Energy balance analysis

Considering the energy in the situation shown in Figure 6.2, the incident harmonics always input energy into the modulated beam, the reflected and transmitted harmonics on the other hand always take energy from the modulated beam. These properties can be easily understood by considering pure sinusoidal or cosine harmonics. However, in many works and also in ours the complex harmonics are used to facilitate the formulation. Complex harmonics also lead to complex energy flux in time domain. This may not cause any problem in frequency domain analysis when cycle averaged values can be obtained by taking the real part of the product of the complex amplitudes of the stress and velocity [100]. In our situation, Equations (6.13) and (6.14) in Section 6.2.3 show that the energy fluxes of the reflected and transmitted waves both may contain contributions from harmonics of different frequencies. Accordingly, it is difficult to obtain a cycle average of the flux using the aforementioned method. To analyze the energy balance, a new strategy is used here.
6.3. Results

Equations (6.12), (6.13) and (6.14) can always be rewritten into the form \( I(t) = |I(t)|e^{i(2\omega t + \varphi_0(t))} \), \( \varphi_0(t) \) is the phase lag. Therefore, at any second the energy input by the incident harmonic and the energy taken by the reflected and transmitted harmonics can be characterized by \( |I_i(t)|, |I_r(t)| \) and \( |I_t(t)| \), respectively. For example, Figure 6.11 shows the time history of the input and scattered energy when a harmonic is incident on a space-only periodic beam within the stop band. In the figure, \( T_e \) is the time period of the incident harmonic, the results are normalized by the maximum input energy during the studied duration. It is well known that within the stop band of a space-only periodic beam, no energy can be transmitted through the beam and the input energy \( |I_i(t)| \) must be equal to the output one \( |I_r(t)| + |I_t(t)| \) to conserve energy. These features are accurately captured in Figure 6.11.

![Figure 6.11](image1)

**Figure 6.11:** Time history of the input, reflected and transmitted energy when a harmonic is incident on a space-only periodic beam within the stop band. The length of the beam is \( L_m = 20\lambda_m \).

For the time-space modulated beam, as shown in Figures 6.6, 6.9 and 6.10, within the first stop band of \( u_0^+ \) mode, when a harmonic is incident from the left side most of the wave will be reflected, but the transmission is not inevitable. Again, these
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

features are well captured by $|I_i(t)|$, $|I_r(t)|$ and $|I_t(t)|$, as shown in Figure 6.12. We can see that the modulated beam absorbs a portion of the incident energy. This feature was also reported by J. Simon [167]. Comparison between his results and ours will be further discussed later in this section. Also we can see that the reflected and transmitted energy fluctuates over time. The fluctuation shows certain periodicity. Indeed, the energy fluxes in Equations (6.13) and (6.14) are still periodic, as proved in Appendix F. Therefore, it is possible to average these energy values within their periods. However, finding their periods requires a lot of mathematical works, especially within a large frequency band. We seek to approximately obtain these mean values in another more easy way. For a periodic function of time $t$, the mean value over a duration will converge to the cycle averaged value as the duration increases. Therefore, we can use the following expression to evaluate the mean power of the incident, reflected and transmitted waves:

$$\text{mean}(|I(\omega)|) = \frac{1}{T_{\text{tot}}} \int_0^{T_{\text{tot}}} |I(t)| dt \quad (6.16)$$

The total integral time $T_{\text{tot}}$ is chosen when doubling it doesn’t change the results.

Figure 6.13 shows the mean power values Vs frequency for three different modulation velocities, a low one ($\beta_m = 0.2$), a moderate one ($\beta_m = 0.4$) and a large one ($\beta_m = 0.6$). All the results in each figure at each frequency are normalized by the corresponding incident mean power. Still we focus our attention within the first stop bands. Concerning the one-way energy insulation, from Figure 6.13 it can be seen that within the first stop band of $u_0^-$ mode, we can obtain very good one-way energy insulation effect using different level of modulation velocity, less than 5% of the incident energy is transmitted in these cases. On the other hand, within the first stop band of the $u_0^+$ mode, for a low modulation velocity ($\beta_m = 0.2$), less than 4% of the incident energy is transmitted. As the modulation velocity increases, the one-way energy insulation becomes less efficient, as already implied by the results in Figures 6.6, 6.9 and 6.10. For a moderate modulation velocity ($\beta_m = 0.4$), the transmitted energy is around 6%, which is still good, but almost 20% of the incident energy will be transmitted when the modulation velocity goes to $\beta_m = 0.6$. Further increase of the modulation velocity will cause more transmission and push the modulated system to the margin of unstable state, which is not recommended.

J. Simon [167] reported that the modulated medium will absorb energy from or input energy into the system. This feature is quite easy to be observed when it is assumed that the incident and reflected waves all only contain a single harmonic. Taking all significant harmonics into account, our results in Figure 6.13 further confirm the above introduced energy balance feature and give more details. Within the stop band of $u_0^-$ mode, when harmonics are incident from the right side against the modulation wave, energy will be given into the system by the modulated beam, this energy is mainly added to the reflected waves. More energy will be inputted by the modulated beam as the modulation velocity increases. On the contrary, within the stop band of $u_0^+$ mode, when harmonics are incident from the left side along the direction of the modulation wave, energy is withdrew from the system. Conse-
6.3. Results

Figure 6.13: Mean power values Vs frequency. All the results in each figure at each frequency are normalized by the corresponding incident mean energy. The length of the modulated beam is $L_m = 20\lambda_m$ in all simulations. Results in the left panel are corresponding to left incident harmonics; results in the right panel are corresponding to right incident harmonics. The modulation parameters are: (a), (b) $\alpha_m = 0.4$, $\beta_m = 0.2$; (c), (d) $\alpha_m = 0.4$, $\beta_m = 0.4$; (e), (f) $\alpha_m = 0.4$, $\beta_m = 0.6$. 
Chapter 6. Reflection and transmission of elastic waves incident on
time-space modulated structures

sequently, the reflected power is significantly reduced, especially for large modulation
velocity. However the total output power \( \text{mean}(|I_r| + |I_t|) \) doesn’t always fall dra-
matically as the modulation velocity increases, because the transmitted power gains
accordingly in this case.

Important conclusions revealed in Figure 6.13 are numerically verified using FE
method. The same FE model shown in Figure 6.3 is used. Tone burst excitations
centred at \( \Omega_e \) are used to generate narrow band incident waves. The incident,
reflected and transmitted waves are measured at points \( O_L \) and \( O_R \) shown in Figure
6.3. For example, when the excitation is applied at the left side of the modulated
beam, the incident and reflected waves are measured at point \( O_L \), the transmitted
waves are measured at point \( O_R \), as shown in Figure 6.14. The energy carried by
these waves are respectively obtained using:

\[
E_{tot} = |S_b \int_{t_1}^{t_2} \sigma(t)v(t)dt|
\]

in which, \( S_b \) is the area of the beam section, \( \sigma(t) \) and \( v(t) \) are respectively the stress
and velocity of the beam section at point \( O_L \) or \( O_R \), \( [t_1, t_2] \) is the time interval
within which one of these three wave groups pass through the measure points.

![Figure 6.14: Waves measured at points \( O_L \) and \( O_R \) in Figure 6.3 when the system
is excited by a tone burst force applied at \( x = 0 \).](image)

We studied the energy balance features of the three different modulated beams
used in Figure 6.13, the results are shown in Figure 6.15. The results are normalized
by the corresponding incident energy in each simulation. In Figure 6.15(a), the
excitation is applied at the left side of the modulated beam, the central frequency
\( \Omega_e \) of the excitation in each simulation is the center of the corresponding \( 1^{st} \) stop
band of \( u_0^+ \) mode. In Figure 6.15(b), the excitation is applied at the right side
of the modulated beam, the central frequency \( \Omega_e \) of the excitation is the center
of the corresponding \( 1^{st} \) stop band of \( u_0^- \) mode in each simulation. For the waves
incident from the left side, from Figure 6.15(a) we can observe good energy insulation
performance when the modulation velocity is low or moderate. Also we can clearly
see the losing of efficiency as the modulation velocity gains. 21.6% of the incident
energy is transmitted when the modulation velocity is \( \beta_m = 0.6 \), which is highly
coincident with the prediction of our theoretical study. When the waves are incident from the right side, Figure 6.15(b) shows that very good energy insulation effects are achieved at different modulation velocity levels. Only 3.2% of the incident energy is transmitted when the modulation velocity is $\beta_m = 0.6$, much less for the low and moderate modulation velocities. Figure 6.15 also clearly shows the energy input and extraction caused by the modulated beam, just as revealed in Figure 6.13.

![Figure 6.15: The incident, reflected and transmitted energy when a tone burst excitation is applied at the left or right side of the modulated beam.](image)

(a) The excitation is at the left side of the modulated beam.

(b) The excitation is at the right side of the modulated beam.

Figure 6.15: The incident, reflected and transmitted energy when a tone burst excitation is applied at the left or right side of the modulated beam. Results are normalized by the corresponding incident energy in each simulation. The central frequency $\Omega_c$ of the excitation in each simulation is illustrated in the figures. The length of the modulated beam is $L_m = 20\lambda_m$ in all the simulations.

### 6.3.6 The feasibility of one-way energy insulation in finite systems

The feasibility of one-way energy insulation in finite systems is discussed in this section. The system composed of three parts shown in Figure 6.16 is considered. The central part is a modulated beam, the rest two parts are uniform beams. The left and right ends of the whole system are respectively free and clamped.

Figure 6.17 shows the spectra of waves in the system at four different instants when the system is excited by a tone-burst force applied at the left end. These results are obtained by using the FE method. The lengths of the uniform and modulated parts are respectively $L_u = 80\lambda_m$ and $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. The spectrum of the force is centred at 0.584, which is within...
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

Figure 6.16: A finite system contains three parts. The left and right parts are uniform beams, the central part is a modulated beam.

the stop band of \( u_0^+ \) mode. In Figure 6.17, the vertical dashed lines indicate the two interfaces at \( x_1 \) and \( x_2 \). Arrows indicate the wave propagation directions.

As already revealed by the scattering matrix (see Figure 6.6), when a harmonic is incident on the modulated beam from the left side at a frequency within the stop band of \( u_0^+ \) mode, most of the incident harmonic will be reflected, also the main frequency will be down-converted. These phenomena are numerically illustrated in the first and second panels in Figure 6.17. Due to the conversion, the frequency of the reflected harmonic by the modulated beam is outside the stop band of \( u_0^+ \) mode. Therefore, when this harmonic is reflected back by the left end of the whole system and is incident on the modulated beam the second time, most of it is transmitted through the modulated beam, as shown in the third and fourth panels in Figure 6.17. Therefore, the expected one-way energy insulation will fail in the finite systems.

The above conclusion is further verified by studying the responses of the system excited by a harmonic load \( F(t) = Le^{i\omega t} \) applied at the left end (see Figure 6.16). The responses are obtained by using the wave-based theory developed in Section 6.2.2. For the system in Figure 6.16, waves \( u^+_m, u^-_m, u^+_s \) and \( u^-_s \) are still expressed by Equations (6.5) and (6.6). However, waves \( u^+_i \) and \( u^-_i \) are unknown now, they should be represented as:

\[
\begin{align*}
  u^+_i(x,t) &= \sum_{q=-R}^{+R} A_q e^{i(\omega + q\omega_m)(t-\frac{x}{c_0})} \\
  u^-_i(x,t) &= \sum_{q=-R}^{+R} G_q e^{i(\omega + q\omega_m)(t+\frac{x}{c_0})}
\end{align*}
\]  
(6.18)

\( A_q \) and \( G_q \) are coefficients to be determined.

These waves need to satisfy not only the continuity conditions at \( x_1 \) and \( x_2 \) but also the boundary conditions at the two ends of the system:

\[
E_0 \frac{\partial [u^+_i(x,t) + u^-_s(x,t)]}{\partial x} \bigg|_{x=0} = Le^{i\omega t}
\]

\[
E_0 \frac{\partial [u^-_i(x_3,t) + u^+_s(x_3,t)]}{\partial x} \bigg|_{x=x_3} = 0
\]

(6.19)
6.3. Results

Figure 6.17: Spectra of waves in the system at different instants. The part between the two vertical dashed lines are the modulated beam. Arrows indicate the wave propagation directions.

Following the process introduced in Section 6.2.2, a set of linear equations are obtained by first substituting the wave expressions (Equations (6.5), (6.6) and (6.18)) into the continuity conditions (6.7) and boundary conditions (6.19), then exploiting the orthogonality of harmonic functions $e^{i(\omega+q\omega_m)t}$. The unknown coefficients of the waves are solved from these equations. The responses of a particular location in the system are then obtained by superposition of corresponding waves. For example, the responses at point $O$ in Figure 6.16 can be expressed as: $u(x_O) = u^+_0(x_O) + u^-_0(x_O)$. In what follows, the responses at this point are studied to verify whether the one-way energy insulation is available in finite systems.

Figure 6.18 shows the responses at point $O$ in a large frequency band. In the simulations, the lengths are $L_u = 10\lambda_m$ and $L_m = 20\lambda_m$. Figure 6.18(b) shows the frequency responses when the central part is a modulated beam with $\alpha_m = 0.4$, $\beta_m = 0.2$. Shadows indicate the stop bands of $u^-_0$ and $u^+_0$ modes. As references, Figure 6.18(a) shows the responses when the central part is only space periodic with $\alpha_m = 0.4$, $\beta_m = 0$. The two stop bands of $u^-_0$ and $u^+_0$ modes are totally overlapping in this case. In these two figures, results corresponding to two truncation orders are shown. They have very good agreement in both figures, which means that the simulations are convergent.
Chapter 6. Reflection and transmission of elastic waves incident on time-space modulated structures

From Figure 6.18(a), it can be seen that within the stop band, the central space-only periodic beam can significantly prevent the energy from being transmitted to the right part. However, when the central beam is time-space modulated, no such effect can be observed within the stop band of $u_0^+$ mode (see Figure 6.18(b)), which verifies that one-way energy insulation fails in finite systems.

![Figure 6.18: Frequency responses at point O shown in Figure 6.16: (a) the central part is only space periodic with $\alpha_m = 0.4$, $\beta_m = 0$; (b) the central part is a modulated beam with $\alpha_m = 0.4$, $\beta_m = 0.2$.](image)

6.4 Conclusions

This chapter concerns reflection and transmission of waves incident on modulated structures. An extended scattering matrix method is developed to study the properties of reflection and transmission. Important conclusions of this chapter are summarized below.

Within the whole stop bands of the two fundamental Bloch modes, approximate one-way wave transmission is available. Also it is demonstrated that, when a single harmonic is incident on the modulated structures, many frequency components will be generated in the reflected and transmitted wave groups. Particularly, the main frequency of the reflected waves is up or down converted, depending on the wave
6.4. Conclusions

incident direction and modulation wave direction. The above unusual phenomena are strongly linked to the Bloch modes in the modulated structures.

The modulation velocity has strong influences on the one-way wave transmission performance. When the harmonic is incident from the left side along the direction of the modulation wave, within the first stop band of $u_0^+$ mode, amplitudes of some harmonics of the transmitted waves increase significantly as the modulation velocity gains. This phenomenon is not observed within the first stop band of $u_0^-$ mode when the harmonic is incident from the right side against the modulation wave.

We also proposed a strategy to study the energy balance of systems containing modulated structures. For low and moderate modulation velocity, modulated structures can achieve very good one-way energy insulation effect within the two first stop bands. The modulated structures will be less efficient in insulating energy within the first stop band of $u_0^+$ mode as the modulation velocity increases. Particularly, almost 20% of the incident energy will be transmitted when the modulation velocity is $\beta_m = 0.6$. The modulated structures input or extract energy from the system. Within the stop band of $u_0^-$ mode, when harmonics are incident against the modulation wave, energy will be inputted into the system. On the contrary, within the stop band of $u_0^+$ mode, when harmonics are incident along the direction of the modulation wave, energy will be extracted from the system. Generally, more larger modulation velocity will lead to more intense energy input or extraction.

The one-way energy insulation can be realized in equivalent infinite or semi-infinite systems but will fail in finite systems. In practice, finite structures can be approximated by infinite ones when waves from a source in them are damped or radiated therefore are not reflected back to the source. Modulated structures can sever as one-way energy insulators in those structures. However, when the finite systems can’t be approximated by infinite ones, the one-way energy insulation will fail due to the frequency conversion phenomenon.
Conclusions and perspectives

In this dissertation, we studied two types of adaptive GRGradient InDex (GRIN) structures. The first one is the piezo-lens. It can focus flexural waves in plates therefore can be used in many applications like energy harvesting and Structural Health Monitoring (SHM). The second type is the time-space modulated structure. It presents a nonreciprocal behavior therefore can be used to build one-way energy insulators. The main contributions of the presented work are summarized below:

- The tunable feature of piezoelectrics shunted with Negative Capacitance (NC) is fully discussed in Chapter 2. The shunted piezoelectric patches are characterized by 5 independent parameters, namely, 2 effective Young’s modulus, 1 effective shear modulus and 2 Poisson’s ratio. Except the effective shear modulus, the other 4 parameters can be tuned by varying the shunting NC values within a large extend. Using these features, the shunted piezoelectric patches are used to build adaptive unit cells. Tunable properties of these cells are demonstrated using an Effective Medium Model (EMM) and the Finite Element Method (FEM), indicating their capability of realizing adaptive GRIN structures.

- An adaptive piezo-lens which can focus flexural waves in plates is designed and studied in Chapter 3. The piezo-lens is composed of piezoelectric patches shunted with NC. Effective refractive indexes inside the piezo-lens are designed to fulfill a hyperbolic secant function by tuning the shunting NC values. It is demonstrated that the piezo-lens can effectively focus waves in a large frequency band. The piezo-lens is efficient in different working conditions. Also the same piezo-lens can focus waves at different designated locations by tuning the NC values.

- Novel harvesting systems are proposed in Chapter 4 for harvesting energy from traveling waves. These systems all contain a piezo-lens and a harvester. The piezo-lens focuses waves at a designed focal point consequently increasing the energy density of the vicinity of that point. Therefore, harvesting at this focal point can yield more energy. Also the piezo-lens can focus waves at different points by tuning the shunting NC values. This feature makes the harvesting system adaptive to the environment changes. The piezo-lens is also combined with Synchronized Switch Harvesting on Inductor (SSH1) based harvesters to significantly enhance the harvested energy from transient waves.

- The free wave propagation in time-space modulated structures and frequency conversion induced by them are studied in Chapter 5. The properties of Bloch modes are first studied. Then, two types of frequency conversion induced by modulated structures are demonstrated and explained. The first type is caused by energy transmission between different orders Bloch modes. The second type
is due to the Bragg scattering effect inside the modulated structures. Due to
the conversion, the frequency can be up or down converted, and the frequency
difference is equal to the modulation frequency.

- The one-wave energy insulation using time-space modulated structures is com-
prehensively evaluated in Chapter 6. Within the whole stop bands of the two
fundamental Bloch modes, approximate one-way wave transmission is avail-
able. Therefore, time-space modulated structures can serve as one-way energy
insulators in equivalent infinite or semi-infinite systems. It is also found that
the one-way energy insulation performance for waves incident along the direc-
tion of the modulation wave becomes less efficient as the modulation velocity
increases. The one-way energy insulation will fail in finite systems due to the
frequency conversion phenomenon.

To further explore the capabilities of adaptive GRIN structures, future attention
is suggested to be paid to the following directions:

- Using the piezo-lens in SHM. As concluded in Chapter 3, the piezo-lens may be
used in energy harvesting and SHM. The former application has been fully
studied in this thesis. The second one is also interesting and remains open.

- Designing other adaptive GRIN structures. As summarized in Chapter 1,
GRIN media were exploited in many applications. Following these examples,
we may also design adaptive GRIN structures to trap waves, even to achieve
cloak effects.

- Developing more advanced circuits. To experimentally study the piezo-lens
and time-space modulated structures or other adaptive GRIN structures, there
are still challenges. The existing circuits that could yield NC all contain
resistive parts which more or less will dissipate energy. They can be used
to soften structures but are potentially not suitable to stiffen them. This fact
limits the realization of adaptive GRIN structures. Future efforts should be
made to develop more advanced circuits (such as synthetic circuits as proposed
in [160]) which have the ability to efficiently soften and stiffen structures.
The Young’s modulus, Poisson’s ratio and density of the aluminum are $E_b = 70 \text{ GPa}$, $
abla_b = 0.3$ and $\rho_b = 2700 \text{ kg/m}^3$, respectively. The material parameters of PZ26 are list in Table A.1.

### Table A.1: Material parameters of PZ26

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{11}^E = S_{22}^E, S_{33}^E$</td>
<td>1.30E-11, 1.96E-11 (Pa$^{-1}$)</td>
<td>Compliance matrix under constant electric field</td>
</tr>
<tr>
<td>$S_{12}^E, S_{13}^E = S_{23}^E$</td>
<td>-4.35E-12, -7.05E-12 (Pa$^{-1}$)</td>
<td></td>
</tr>
<tr>
<td>$S_{44}^E = S_{55}^E, S_{66}^E$</td>
<td>3.32E-11, 3.47E-11 (Pa$^{-1}$)</td>
<td></td>
</tr>
<tr>
<td>$d_{31} = d_{32}$</td>
<td>-1.28E-10 (C/N)</td>
<td>Piezoelectric matrix</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>3.28E-10 (C/N)</td>
<td></td>
</tr>
<tr>
<td>$d_{24} = d_{15}$</td>
<td>3.27E-10 (C/N)</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>7700 (kg/m$^3$)</td>
<td>Density</td>
</tr>
<tr>
<td>$\varepsilon_{11}^\sigma = \varepsilon_{22}^\sigma, \varepsilon_{33}^\sigma$</td>
<td>$1.19E + 03\varepsilon_0, 1.33E + 03\varepsilon_0$</td>
<td>Dielectric permittivity under constant stress</td>
</tr>
</tbody>
</table>
Appendix B

Fully coupled finite element models of piezoelectric systems and Bloch boundary conditions

Finite element models of piezoelectric systems

A generic piezo-mechanical system is illustrated in Figure B.1. In this system, the mechanical structure occupies a domain $\Omega_m$ with particular Dirichlet boundary conditions applied on the surface $S_m^u$ and Neumann boundary conditions on the surface $S_m^\sigma$. A set of piezoelectric transducers are connected to the mechanical structure occupying a domain $\Omega_e$ (only one is depicted in the figure as an example). Zero charge conditions are applied to the lateral surfaces $S_e^l$ of the piezoelectric transducers. The bonding interfaces $S_{m-e}$ are grounded and the free electrode $S_e^i$ of each piezoelectric transducer is applied with a surface charge $Q_i$ or a voltage $V_i$, here $i \in [1,2,...,P]$ and $P$ is the total number of the piezoelectric transducers.

![Figure B.1: A generic piezo-mechanical system.](image)

The 3D dynamical equilibrium equations for the piezo-mechanical system described above can be written as:

$$\rho \ddot{w} - \nabla \cdot \sigma = f, \quad \forall x \in \Omega_m \cup \Omega_e$$
$$\nabla \cdot D = 0, \quad \forall x \in \Omega_e$$

(B.1)

with associated mechanical boundary conditions:
Appendix B. Fully coupled finite element models of piezoelectric systems and Bloch boundary conditions

\[ w = w_0, \quad \forall x \in S^u_m \]
\[ \sigma \cdot n = T_0, \quad \forall x \in S^\sigma_m \]  
\[ (B.2) \]

and electric boundary conditions:

\[ D \cdot n = 0, \quad \forall x \in S^l_e \]
\[ \varphi = 0, \quad \forall x \in S_{m-e} \]
\[ \int D \cdot n \, ds = -Q_i \quad \text{or} \quad \varphi = V_i, \quad \forall x \in S^l_e \quad \text{and} \quad i \in [1, 2, ..., P] \]  
\[ (B.3) \]

In the equations above, \( w \) is the mechanical displacement tensor, \( \sigma \) is the stress tensor, \( f \) is the applied external force tensor, \( D \) is the electric displacement tensor, and \( \varphi \) is the electric potential, \( n \) is the outward unit normal vector.

The stress tensor and the electric displacement tensor in piezoelectric materials are related to the linear strain tensor \( \varepsilon \) and electric field tensor \( E \) through the constitutive relations below:

\[ \sigma = C_E : \varepsilon - e^T \cdot E \]
\[ D = e \cdot \varepsilon + \varepsilon_S \cdot E \]  
\[ (B.4) \]

here, \( C_E, e \) and \( \varepsilon_S \) denote the elasticity tensor at constant electric field, the piezoelectric coupling tensor and the dielectric permittivity tensor at constant strain, respectively. \( (\cdot)^T \) indicates transposition. The strain tensor is calculated as \( \varepsilon = \frac{1}{2} \left( \nabla \cdot w + \nabla^T \cdot w \right) \). The electric field tensor and the electric potential are related as \( E = -\nabla \varphi \).

Using the above equations and finite element discretization technique, the discrete governing equations for the piezo-mechanical system are obtained [113, 168]:

\[ M_{dd} \ddot{d} + K_{dd} d + K_{dV} V = F \]
\[ -K_{dV}^T \dot{d} + K_{VV} V = Q \]  
\[ (B.5) \]

here, \( d \) and \( V \) represent the structural and voltage Degree Of Freedoms (DOFs), respectively; \( F \) and \( Q \) are the mechanical forces and charges flowing into the piezoelectric patch, respectively. Details of \( M_{dd}, K_{dd}, K_{dV} \) and \( K_{VV} \) as well as \( F \) and \( Q \) can be found in [113, 168].

The potential DOFs in the piezoelectric patches are partitioned into DOFs inside the patches \( V_i \), DOFs on the free electrodes of the patches \( V_g \) and DOFs on the bonding surfaces \( V_g \). The potential DOFs on the bonding surfaces are grounded, thus the corresponding equations and columns are directly removed from the equations. Consequently, the governing equations in (B.5) are partitioned as below:
\[
\begin{bmatrix}
M_{dd} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\ddot{d} \\
\dot{V}_i \\
\dot{V}_p
\end{Bmatrix}
+ \begin{bmatrix}
K_{dd} & K_{di} & K_{dp} \\
-K_{di}^T & K_{ii} & K_{ip} \\
-K_{dp} & -K_{ip}^T & K_{pp}
\end{bmatrix}
\begin{Bmatrix}
\dot{d} \\
\dot{V}_i \\
\dot{V}_p
\end{Bmatrix}
= \begin{Bmatrix}
F \\
Q_i \\
Q_p
\end{Bmatrix}
\] (B.6)

Since there is no charge source inside the piezoelectric patches \((Q_i = 0)\), the internal DOFs \(V_i\) can be determined by exact static condensation from Equation (B.6):

\[
V_i = -K_{ii}^{-1}(-K_{ii}^T d + K_{ip} V_p)
\] (B.7)

After the condensation, the system equations become:

\[
\begin{bmatrix}
M_{dd} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\ddot{d} \\
\dot{V}_p
\end{Bmatrix}
+ \begin{bmatrix}
G_{dd} & G_{dp} \\
-G_{dp}^T & G_{pp}
\end{bmatrix}
\begin{Bmatrix}
\dot{d} \\
\dot{V}_p
\end{Bmatrix}
= \begin{Bmatrix}
F \\
Q_p
\end{Bmatrix}
\] (B.8)

with

\[
G_{dd} = K_{dd} + K_{di}K_{ii}^{-1}K_{di}^T
\]
\[
G_{dp} = K_{dp} - K_{di}K_{ii}^{-1}K_{ip}
\]
\[
G_{pp} = K_{pp} + K_{ip}K_{ii}^{-1}K_{ip}
\] (B.9)

As the DOFs on one electrode have identical potentials, the potential DOFs on the free electrode of each piezoelectric patch are reduced such that only one master potential DOF remains per patch. The reduction is achieved by using an explicit transformation:

\[
V_p = T_m V_i
\] (B.10)

here, \(T_m\) is the transformation matrix, it represents the Null space of the linear transformation in Equation (B.10), the nonzero terms in it is either 1 or -1.

Substituting Equation (B.10) into Equation (B.8) and then left multiplying the transposed transformation matrix yield the final fully coupled governing equations:

\[
\begin{bmatrix}
M_{dd} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\ddot{d} \\
\dot{V}_m
\end{Bmatrix}
+ \begin{bmatrix}
H_{dd} & H_{dm} \\
-H_{dm}^T & C_m
\end{bmatrix}
\begin{Bmatrix}
\dot{d} \\
\dot{V}_m
\end{Bmatrix}
= \begin{Bmatrix}
F \\
Q_m
\end{Bmatrix}
\] (B.11)

with

\[
H_{dd} = G_{dd}, \quad H_{dp} = G_{dp}T_m
\]
\[
C_m = T_m^T G_{pp} T_m, \quad Q_m = T_m^T Q_p
\] (B.12)

The shunting circuit with impedance \(Z^{su}\) can be easily introduced into the piezoelectric system represented by Equation (B.11) through:

\[
\dot{Q}_m(i) = -\frac{V_m(i)}{Z^{su}}
\] (B.13)

here, \(i\) indicates the \(i^{th}\) piezoelectric patch in the system.
Appendix B. Fully coupled finite element models of piezoelectric systems

Fully coupled finite element models of piezoelectric systems

Bloch boundary conditions

The Finite Element Method (FEM) and Bloch Boundary Conditions (B.C) were widely used to calculate dispersion curves of periodic structures. It was also used to obtain the dispersion relations of the structure composed of cells containing shunted piezoelectric patches shown in Figure 2.5 in Section 2.4 of Chapter 2. The FE model of the cell is illustrated in Figure B.2. The lower piezoelectric patch and the shunts are not shown in the figure. The mechanical DOFs on the four lateral surfaces of the host plate are respectively $d_1$, $d_2$, $d_3$ and $d_4$, as indicated in the figure. The B.C are applied on these surfaces through the relations [169, 170]:

$$d_2 = d_1 e^{-iklb}, \quad d_4 = d_3 e^{-iklb}$$

(B.14)

in which, $l_b$ is the width and length of the unit cell; the wave vector is defined as $k = k_x e_1 + k_y e_2$, $e_1$ and $e_2$ denote the unit vectors along the $x$ and $y$ axes, respectively. Using Equations (B.11), (B.13) and (B.14), a well-posed eigenvalue problem can be obtained [169], solving which by varying the wave vectors $k$ along the boundaries of the first irreducible Brillouin zone results in the dispersion curves and corresponding modes.

![Figure B.2](image_url)

Figure B.2: FE model of the cell containing shunted piezoelectric patches. The lower patch and the shunts are not shown in the figure.
Appendix C

Supplementary results for Chapter 3

Ideal lens design

A flat GRIN lens to focus flexural waves in thin plate can be obtained if the refractive index for flexural wave inside the lens zone fulfills a hyperbolic secant function [16]:

\[ n(y) = n_0 \cdot \text{sech}[\alpha(y - \beta)] \] (C.1)

in which, \( n_0 \) represents the refractive index of the background plate, \( \alpha \) is the gradient coefficient and \( \beta \) represents the \( y \) coordinate of the symmetry axis of the refractive index profile. Waves incident into the lens from the \( x \) direction will be focused at a focal point on the \( y = \beta \) line, with a focal length represented as \( f = \pi/2\alpha \).

The refractive index for flexural waves incident from the background plate into the GRIN lens is defined as the ratio of the phase velocity in the background plate to the phase velocity in the lens:

\[ n(y) = \frac{c_b}{c(y)} \] (C.2)

At sub-wavelength frequency band, Equation (C.2) can be further written as:

\[ n(y) = \left\{ \frac{\rho(y)[1 - \mu^2(y)]}{E(y)\bar{h}(y)} \cdot \frac{E_b h_b}{\rho_b(1 - \mu_b^2)} \right\}^{1/4} \] (C.3)

According to Equation (C.3), it can be observed that the refractive index profile illustrated in Equation (C.1) can be realized by varying a single material parameter or the thickness of the lens, or even a combination of several of these parameters.

Assume that the thickness, Poisson’s ratios and densities of the plate and the lens are all identical. Therefore, a GRIN lens can be realized by consecutively varying the Young’s modulus inside the lens zone according to:

\[ E(y) = E_b \cdot \text{sech}^{-4}[\alpha(y - \beta)] \] (C.4)

Since the refractive index profile inside this GRIN lens zone is perfectly fulfilled, this lens is termed ideal lens.
Numerical results

The numerical studies were performed by using finite element method. In the simulations, the structures were discretized by 3D quadratic Lagrange elements. At least 10 elements were guaranteed in one wavelength. The simulation domain was surrounded by perfectly matched layers to avoid wave reflections at the boundaries [129, 130]. The ideal lens has dimensions of \(0.24\ m \times 0.58\ m\) in the \(x-y\) plane, as depicted in Figure C.1. The thickness of the plate and the ideal lens are \(0.005\ m\). The background plate is made of aluminum with \(E_b = 70\ Gpa\), \(\mu_b = 0.3\) and \(\rho_b = 2700\ kg/m^3\). The Poisson’s ratio and density of the ideal lens are identical with the plate. The Young’s modulus of the lens satisfies Equation (C.4). Light damping amount to a hysteresis coefficient of 0.1% was applied. The parameters of ideal lens were set as \(\alpha = \pi/0.6\) and \(\beta = 0\). With these settings, theoretically flexural waves will be focused at a distance \(f = 0.3\ m\) on the \(y = 0\) line.

![Figure C.1: Ideal lens on plate](image)

Figure C.2 shows the normalized power flows at 2000 Hz when flexural waves are incident into the ideal lens in the \(x\) direction. It can be observed that most of the power inside the lens is flowing toward the designed focal point. But near the upper and lower boundaries of the lens, the power flows diverge from the designed paths. These results demonstrate that the GRIN lens designed according to Equation (C.1) has weak bending effect for the waves incident near the upper and lower boundaries.

Figure C.3 shows the normalized kinetic energy distribution at 6000 Hz when plane waves are incident into the ideal lens with an angle equal to \(20^\circ\). A second focus can be observed in the bottom left direction of the main focus. The occurrence of the second focus can be interpreted. The lens focuses obliquely incident plane waves upward the designed location. Most of the waves be focused are travelling from the bottom left direction, they will be more concentrated as they approach the focal zone, therefore, peaks may be observed near the focal zone.

For the piezo-lens, at lower frequencies, the second focuses are inside the lens zone, it is difficult to recognize them since the wave field inside the piezol-lens is complicated. The focal zone of the piezo-lens shifts rightward as the frequency
Figure C.2: Normalized power flows at 2000 Hz when plane waves are incident into the ideal lens in the $x$ direction. Black cross indicates the designed focal point.

Figure C.3: Normalized kinetic energy distribution at 6000 Hz when plane waves are incident into the ideal lens with an angle equal to 20°.

increases. Thus, at higher frequency the second focus shifts outside the lens and becomes visible.
In Chapter 4, we developed a Corrected Reduced Model (CRM) to study the piezoelectric system in Figure D.1(a). Accuracy of this CRM is validated in this appendix.

Models

The Full finite element Model (FM) of the system in time domain is described by equations:

\[ M_{dd}\ddot{d} + H_{dd}d + H_{dV}V = F \]  
\[ -H_{dV}^T\dot{d} + C_{Lh}\dot{V} = \dot{Q} \]

in which:

\[ H_{dV} = [H_{dL}, H_{dh}], \ V = [V_L, V_h]^T \]

\[ Q = [Q_L, Q_h]^T, \ C_{Lh} = \begin{bmatrix} C_L & 0 \\ 0 & C_h \end{bmatrix} \]

here, matrices or vectors with subscript \( L \) are related to piezoelectric patches in the piezo-lens, those with subscript \( h \) are linked to the patch composing the harvester. The matrices \( C_L \) and \( C_h \) are diagonal, each of their diagonal elements represents the blocked intrinsic capacitance of a piezoelectric patch; \( V_L, V_h \) are the master DOFs on the free electrodes of the patches; \( Q_L, Q_h \) are the charges flowing to the patches.

Using the truncated mode bases of the system when all the piezoelectric patches are short-circuit (namely, \( V = 0 \)), we can reduce the FM:

\[ \ddot{\eta} + \Lambda \eta + \Phi^T H_{dV} V = \Phi^T F \]  
\[ -H_{dV}^T \Phi \dot{\eta} + C_{Lh} \dot{V} = \dot{Q} \]

in which, \( \Phi = [\phi_1, \phi_2, ..., \phi_m] \), \( \Lambda = diag(\omega_i^2) \) and \( \eta = \Phi^{-1}d \). \( \phi_i \) and \( \omega_i \) are respectively the \( i^{th} \) natural mode and frequency of the short-circuit piezoelectric system.
Appendix D. Validation of the corrected reduced models of piezoelectric systems

Figure D.1: (a) The simulation mode and (b) the applied tone burst excitation.

We call the model in Equation (D.3) the Original Reduced Model (ORM). It is reported in several papers [128, 107] that this ORM can’t accurately describe the piezoelectric effects of the system. Therefore, in Chapter 4, we corrected the ORM by modifying the capacitance matrix:

\[ \ddot{\eta} + \Lambda \dot{\eta} + \Phi^T H_{dv} V = \Phi^T F \]  
\[ -H_{dv}^T \Phi \dot{\eta} + C_{Lh}^* \dot{V} = \dot{Q} \]

in which,

\[ C_{Lh}^* = \begin{bmatrix} C_L^* & 0 \\ 0 & C_h^* \end{bmatrix} = \text{diag}(H_{dv}^T(H_{dd}^{-1} - \Phi \Lambda^{-1} \Phi^T)H_{dv}) + C_{Lh} \]

Model in Equation (D.4) is the CRM. In the following section, the FM, ORM and CRM are all used to study the dynamic responses of the system in Figure D.1(a), results of them are compared.
Validation

The system in Figure D.1(a) is used to harvest energy from transient waves in Chapter 4. Our interest is to check whether the CRM can capture the piezoelectric behavior of the system, consequently to give accurate output of the piezoelectric patch for harvesting.

In the simulations for validation, a 10-period Hanning-windowed tone burst excitation centred at frequency 2000 Hz is applied at the left end of the plate to generate transient waves. The excitation is a line transverse force, its waveform and spectrum are illustrated in Figure D.1(b). Modes that have natural frequencies smaller than five times of the maximum frequency of interest are remained in the reduced model.

We have done two groups of simulations. In each group, the system is respectively studied by using the FM, ORM and CRM. In the first group, the piezo-lens is disabled by letting all the piezoelectric patches being open-circuit. In the second group, the piezo-lens is active, namely piezoelectric patches in it are connected with NC circuits. Parameters of the lens are chosen as $\alpha = \pi$, $\beta = 0$. In both the two groups, the piezoelectric patch composing the harvester is open-circuit.

Figure D.2 shows the results of the two simulation groups. The responses are measured at the left-bottom corner on the upper surface of the piezoelectric patch for harvesting. From Figure D.2(a) we can see that both the ORM and CRM can obtain accurate mechanical response at the measure point. However, the voltage output from ORM has significant errors up to 30% at some instants. On the contrary, the CRM can give accurate voltage response. The advantage of the CRM compared with the ORM is further revealed in Figure D.2(b). We found that, when the piezo-lens is active (i.e., the piezoelectric patches in the lens are connected with NC circuits), the ORM becomes unstable therefore its results are not shown in Figure D.2(b). The CRM still has very good accuracy to predict both the mechanical and electrical responses. Also compared with the FM, the CRM significantly reduces the simulation time from hours to minutes.
Appendix D. Validation of the corrected reduced models of piezoelectric systems

(a) All piezoelectric patches are open-circuit.

(b) The piezo-lens is active with designed parameters $\alpha = \pi$, $\beta = 0$, the piezoelectric patch for harvesting is open-circuit.

Figure D.2: Comparison of the mechanical and electrical responses of the left-bottom corner on the upper surface of the piezoelectric patch for harvesting. CRM: Corrected Reduced Model, ORM: Original Reduced Model, FM: Full Model.
Details of the matrices in Chapter 6:

\( \mathbf{M}_{A_1} = \text{diag}(e^{-i \omega m x_1}) \); \( \mathbf{M}_{B_1} = \mathbf{M}_{A_1}^{-1} \)

\( \mathbf{M}_{A_2} = \text{diag}(E_0 + q \omega m e^{-i \omega m x_1}) \); \( \mathbf{M}_{B_2} = \mathbf{M}_{B_1} \mathbf{M}_{A_2} \mathbf{M}_{B_1} \)

\( \mathbf{M}_{F_3} = \text{diag}(e^{-i \omega m x_2}) \); \( \mathbf{M}_{G_3} = \mathbf{M}_{F_3}^{-1} \)

\( \mathbf{M}_{F_4} = \text{diag}(E_0 + q \omega m e^{-i \omega m x_2}) \); \( \mathbf{M}_{G_4} = \mathbf{M}_{G_3} \mathbf{M}_{F_4} \mathbf{M}_{G_3} \)

\[ \mathbf{M}_{C_1}(q + R + 1, n + R + 1) = U^+(n,q) e^{-i(k_n^+ + qk_m)x_1} \]

\[ \mathbf{M}_{D_1}(q + R + 1, n + R + 1) = U^-(n,q) e^{-i(k_n^- + qk_m)x_1} \]

\[ \mathbf{M}_{C_2}(q + R + 1, n + R + 1) = \sum_{p=-1}^{+1} \hat{E}_p e^{-ipkmx_1} [(k_n^+ + (q - p)k_m)U^+(n,q-p)e^{-i(k_n^+ + (q-p)k_m)x_1}] \]

\[ \mathbf{M}_{D_2}(q + R + 1, n + R + 1) = \sum_{p=-1}^{+1} \hat{E}_p e^{-ipkmx_1} [(k_n^- + (q - p)k_m)U^-(n,q-p)e^{-i(k_n^- + (q-p)k_m)x_1}] \]

\[ \mathbf{M}_{C_3}(q + R + 1, n + R + 1) = U^+(n,q) e^{-i(k_n^+ + qk_m)x_2} \]

\[ \mathbf{M}_{D_3}(q + R + 1, n + R + 1) = U^-(n,q) e^{-i(k_n^- + qk_m)x_2} \]

\[ \mathbf{M}_{C_4}(q + R + 1, n + R + 1) = \sum_{p=-1}^{+1} \hat{E}_p e^{-ipkmx_2} [(k_n^+ + (q - p)k_m)U^+(n,q-p)e^{-i(k_n^+ + (q-p)k_m)x_2}] \]

\[ \mathbf{M}_{D_4}(q + R + 1, n + R + 1) = \sum_{p=-1}^{+1} \hat{E}_p e^{-ipkmx_2} [(k_n^- + (q - p)k_m)U^-(n,q-p)e^{-i(k_n^- + (q-p)k_m)x_2}] \]

in which, \( n, q = -R, ..., 0, ... R \).

Details of the matrices in Equations (6.10) in Chapter 6:

\[ \mathbf{H}_{BF_{11}} = \mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{B_1} \]

\[ - (\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{D_1} - \mathbf{M}_{D_3})(\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{D_1} - \mathbf{M}_{D_3})^{-1}(\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{B_1} + \mathbf{M}_{B_2}) \]

\[ \mathbf{H}_{BF_{21}} = \mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{B_1} \]

\[ - (\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{D_1} - \mathbf{M}_{D_3})(\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{D_1} - \mathbf{M}_{D_3})^{-1}(\mathbf{M}_{C_3} \mathbf{M}_{C_1}^{-1} \mathbf{M}_{B_1} + \mathbf{M}_{B_2}) \]

\[ \mathbf{H}_{BF_{12}} = -\mathbf{M}_{F_3}; \quad \mathbf{H}_{BF_{22}} = -\mathbf{M}_{F_4} \]
\[ H_{AG_{11}} = -M_{C_3}M_{C_1}^{-1}M_{A_1}I_0 \]
\[ + (M_{C_3}M_{C_1}^{-1}M_{D_1} - M_{D_3}) (M_{C_2}M_{C_1}^{-1}M_{D_1} - M_{D_2})^{-1} (M_{C_2}M_{C_1}^{-1}M_{A_1} - M_{A_2}) I_0 \]
\[ H_{AG_{21}} = -M_{C_4}M_{C_1}^{-1}M_{A_1}I_0 \]
\[ + (M_{C_4}M_{C_1}^{-1}M_{D_1} - M_{D_4}) (M_{C_2}M_{C_1}^{-1}M_{D_1} - M_{D_2})^{-1} (M_{C_2}M_{C_1}^{-1}M_{A_1} - M_{A_2}) I_0 \]
\[ H_{AG_{12}} = M_{G_3}I_0; \ H_{AG_{22}} = -M_{G_4}I_0 \]
Appendix F

Proof of the periodicity of Equations (6.13) and (6.14) in Chapter 6

Recall that Equation (6.13) is:

\[ I_r(t) = -E_0 \sum_{q_1=-R}^{+R} \sum_{q_2=-R}^{+R} B_{q_1} B_{q_2} (\omega + q_1 \omega_m)(\omega + q_2 \omega_m) e^{2i(\omega + \frac{q_1 + q_2}{2} \omega_m)(t + \frac{c_1}{c_0})} \]  

(F.1)

And Equation (6.14) is:

\[ I_t(t) = E_0 \sum_{q_1=-R}^{+R} \sum_{q_2=-R}^{+R} F_{q_1} F_{q_2} (\omega + q_1 \omega_m)(\omega + q_2 \omega_m) e^{2i(\omega + \frac{q_1 + q_2}{2} \omega_m)(t - \frac{c_2}{c_0})} \]  

(F.2)

They both are superposition of harmonics, whose periods are \(|2\omega + (q_1 + q_2)\omega_m|\), in which \(\omega\) and \(\omega_m\) are rational numbers; \(q_1\) and \(q_2\) are integers belong to the set \([-R, R]\).

Assume that \(f(x)\) and \(g(x)\) are two periodic functions. Periods of them are \(F\) and \(G\), respectively. The sum of these two periodic functions remains periodic if the common multiple \(C_{FG}\) of \(F\) and \(G\) exists and \(C_{FG}\) is the period of the new function, namely \(f(x) + g(x) = f(x + C_{FG}) + g(x + C_{FG})\). Accordingly, Equations (F.1) and (F.2) are periodic if the common multiple of all the elements in the set \(|2\omega + (q_1 + q_2)\omega_m|\) exist. In what follows, we will prove that we can always find the above mentioned common multiple for any rational \(\omega\) and \(\omega_m\).

It is easy to understand that if the common multiple of any two arbitrary elements in the set \(|2\omega + (q_1 + q_2)\omega_m|\) exist, then the common multiple of all the elements in the set exist. For simplicity, we use two positive numbers \(a + mb\) and \(a + nb\) to represent two arbitrary elements in the set. Therefore, \(a\) and \(b\) are positive rational numbers; \(m\) and \(n\) are integers and \(m \neq n\). We assume that the common multiple of \(a + mb\) and \(a + nb\) is \(c\), and we have:

\[ c = c_1(a + mb), \; c = c_2(a + nb) \]  

(F.3)

The question is whether we can find these two positive integers \(c_1\) and \(c_2\)? The answer is yes!

From Equation (F.3) we have:

\[ \frac{c_1 - c_2}{c_2 m - c_1 m} = \frac{b}{a} \]  

(F.4)
Appendix F. Proof of the periodicity of Equations (6.13) and (6.14) in Chapter 6

Since $a$ and $b$ are positive rational, we can always find two positive integers $A$ and $B$ to satisfy:

\[ \frac{b}{a} = \frac{(n-m)B}{(n-m)A} \]  \hspace{1cm} (F.5)

Recall that $m \neq n$.

According to Equations (F.4) and (F.5) we can obtain two equations:

\[ c_1 - c_2 = (n-m)B, \quad c_2n - c_1m = (n-m)A \]  \hspace{1cm} (F.6)

Solving Equations (F.6) we have:

\[ c_1 = A + nB, \quad c_2 = A + mB \]  \hspace{1cm} (F.7)

$A + nB = \frac{A}{a}(a + nb) > 0$ and $A + mB = \frac{A}{a}(a + mb) > 0$, therefore, $c_1$ and $c_2$ are all positive integers, which means the common multiple $c$ of any two arbitrary positive numbers $a + mb$ and $a + nb$ exist. Consequently, it proves that Equations (F.1) and (F.2) are periodic.
## List of abbreviation

<table>
<thead>
<tr>
<th>Short form</th>
<th>Signification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>One dimension</td>
</tr>
<tr>
<td>2D</td>
<td>Two dimension</td>
</tr>
<tr>
<td>3D</td>
<td>Three dimension</td>
</tr>
<tr>
<td>ABC</td>
<td>Absorption Boundary Condition</td>
</tr>
<tr>
<td>ABH</td>
<td>Acoustic Black Hole</td>
</tr>
<tr>
<td>AC</td>
<td>Alternating Current</td>
</tr>
<tr>
<td>SSH</td>
<td>Adaptive Synchronized Switch Harvesting</td>
</tr>
<tr>
<td>B.C</td>
<td>Bloch boundary Conditions</td>
</tr>
<tr>
<td>CRM</td>
<td>Corrected Reduced Model</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree Of Freedom</td>
</tr>
<tr>
<td>DSSH</td>
<td>Double Synchronized Switch Harvesting</td>
</tr>
<tr>
<td>ESSH</td>
<td>Enhanced Synchronized Switch Harvesting</td>
</tr>
<tr>
<td>EMM</td>
<td>Effective Medium Model</td>
</tr>
<tr>
<td>FBZ</td>
<td>First Brillouin Zone</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>FM</td>
<td>Full Model</td>
</tr>
<tr>
<td>GRIN</td>
<td>GGradient INdex</td>
</tr>
<tr>
<td>IBZ</td>
<td>Irreducible Brillouin Zone</td>
</tr>
<tr>
<td>NC</td>
<td>Negative Capacitance</td>
</tr>
<tr>
<td>ORM</td>
<td>Original Reduced Model</td>
</tr>
<tr>
<td>PC</td>
<td>Phononic Crystal</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>PVDF</td>
<td>Polyvinylidene Fluoride</td>
</tr>
<tr>
<td>PWE</td>
<td>Plane Wave Expansion</td>
</tr>
<tr>
<td>PZT</td>
<td>Lead-Zirconate-Titanate</td>
</tr>
<tr>
<td>QEP</td>
<td>Quadratic Eigenvalue Problem</td>
</tr>
<tr>
<td>SHM</td>
<td>Structural Health Monitoring</td>
</tr>
<tr>
<td>SSHI</td>
<td>Synchronized Switch Harvesting on Inductor</td>
</tr>
</tbody>
</table>
List of Figures

1.1 A hyperbolic secant refractive index profile (left) along the direction transverse to propagation enables redirection of incident waves inside the medium (right) [16]. 2
1.2 A GRIN lens is obtained by (left) adjusting radii of cylinders or (right) changing elastic properties of cylinders along the transverse direction [16]. 3
1.3 One cell of the metamaterial for GRIN lens. The top part is a lead disc, the central part is a silicone rubber and the low part is an aluminum plate [6]. 3
1.4 (a) Details of the metamaterial array, (b) Close-up 7 × 9 planar metamaterial array region and trajectory of Lamb waves focusing [6]. 4
1.5 Scheme of a circular flexural lens with radius $R_s$ and a radial dependent thickness $h(r)$ to achieve the desired refractive index $n(r)$ [26]. 4
1.6 The metasurface and schematics for the derivation of the acoustic focusing with focus length $f$ [28]. 5
1.7 Plate with a power-law tapered edge. Ideally, the plate should extend to $x = 0$ with a zero thickness at that point. However, due to the inevitable truncation in practice, the plate is truncated at $x = x_1$ [31]. 6
1.8 The elastic wedge of power-law profile studied in [30]. 7
1.9 (a) A circular ABH on a plate [29]. (b) Typical ray trajectories illustrating propagation of flexural waves over a circular ABH [39]. 8
1.10 (a) Sketch of an infinite Euler-Bernoulli beam with periodic ABH cells with a lattice constant $a$ [44]. (b) Schematic of the phononic thin plate with a square ABH periodic lattice structure [43]. 9
1.11 (a) Schematic view of the manufactured Linear Acoustic Black Hole (LABH) showing the wooden backing and the distribution of the ribs whose inner radii decrease to almost zero. (b) Prototype of the LABH used in experiments [46]. 10
1.12 Plate with an ABH of an Archimedean spiral shape [47]. 10
1.13 Photograph of the structure studied in [49]. The outer shell is made of cylinders whose diameters increase with decreasing distance to the center. The inner core is made of identical cylinders in a hexagonal lattice with about 84% of filling fraction. The inset shows the ray trajectories of the sound traveling within the outer shell. 11
1.14 Fan blade profile with tapering according to the power-law geometry [51]. 11
1.15 Schematic models for tapered plate with shunted PZT [52]. 12
1.16 Simulation domain for a horizontally incident acoustic plane wave. 17
1.17 (a) Pressure field without obstacle. (b) Pressure field with a bare obstacle, circle dash line indicates the obstacle. (c) Pressure field with an obstacle surrounded by an ideal cloak, circle dash lines indicate the boundaries of the cloak. (d) Pressure field with an obstacle surrounded by an approximate cloak, circle dash lines indicate the boundaries of the cloak. ................................................................. 18

1.18 Pressures along the central line $y = 0$. Red line, blue dot-dash line and green dot-dash line represent the case without obstacle, the case with an obstacle surrounded by an ideal cloak and the case with an obstacle surrounded by an approximate cloak, respectively. Vertical dot-dash lines represent inner boundaries of the cloaks, vertical dash lines represent outer boundaries of the cloaks. ................................................................. 19

1.19 One-dimensional harmonic mass-spring system with spatial and temporal sinusoidal modulation of the spring stiffness: $\beta(x, t)$ as a realization of an elastic time-dependent superlattice. The spatial modulation propagates in time with the velocity $\pm V$ [80]. .................. 24

1.20 Illustration of reciprocity in conventional media. $S$ is the source and $D$ means the detector. ................................................................. 26

1.21 Band structure of the longitudinal modes in a beam with time-space modulated Young’s modulus [93]. $\mu$ and $\Omega$ are dimensionless wavenumber and frequency, respectively. .................. 27

1.22 (a) Schematic of a $2L$ long time-space modulated beam loaded in its mid-span, longitudinal motion is excited by the horizontal force $F_{\text{ext}, L}$. (b) Forced wave propagation in the beam in Figure 1.22(a) when the frequency of the excitation is within the left stop band in Figure 1.21 (left) and when the frequency of the excitation is within the right stop band in Figure 1.21 (right). These results are from [93]. 28

2.1 A typical piezoelectric material with the top and bottom surfaces electrode and $x3$ axis in the polling direction [2]. $C_{\text{neg}}$ indicates the shunted NC circuit. ................................................................. 30

2.2 Actuation modes of piezoelectric actuators, $P$ indicates the direction [98]. ................................................................. 32

2.3 Real-life negative capacitance circuit used in F. Tateo et al.’s works [100, 101]. ................................................................. 33

2.4 Variation of the (a) effective Young’s moduli and (b) effective Poisson’s ratios with respect to the $C_{\text{neg}}$. ................................................................. 36

2.5 Top and side views of the elementary cell containing piezoelectric patches shunted with NC circuits. ................................................................. 37

2.6 Variation of the effective bending stiffness with the $C_{\text{neg}}$. Red stars indicate the $C_{\text{neg}}$ values used in the simulations in Section 2.4.2. ................................................................. 38
2.7 Dispersion relations of the waves in structures composed of cells shown in Figure 2.5 corresponding to different $C_{neg}$ values. Dispersion curves of the flexural mode (i.e., the $A_0$ mode) obtained by using the EMM are also shown in these figures. Shadows indicate the 1st band gaps of the $A_0$ mode along the $x$ direction. 39

2.8 (a) A structure with infinitely repeating unit cells, (b) a single unit cell containing shunted piezoelectric patches [110]. 40

2.9 A hybrid media composed of a mechanical substrate and an piezoelectric elements interconnected through inductive elements [124]. 41

2.10 Unit cell with its characteristic dimensions; the shaded regions correspond to piezoelectric patches. $Z^{SH}$ is the equivalent electrical impedance of the shunting circuit [4]. 41

2.11 (a) Schematic of the GRIN metamaterial-enhanced flexural wave sensing system, (b) an aluminum beam is bonded with a periodic array of piezoelectric (PZT-5A) patches, which are shunted by a gradient array of NC circuits to realize an adaptive GRIN metamaterial waveguide [126]. 42

3.1 (a) The harvesting system with piezo-lens and (b) the gradient variation profile of the refractive index $n(y)$. 46

3.2 Top and side view of one cell in the piezo-lens. 46

3.3 Variation of effective refractive index with negative capacitance value. 47

3.4 Normalized power flows in the host plate at 2000 Hz (a) without and (b) with piezo-lens. Black cross indicates the designed focal point. 50

3.5 Normalized kinetic energy distribution after the piezo-lens at 2000 Hz. Black cross indicates the designed focal point, solid line indicates the energy concentration zone. 51

3.6 Energy enhancement ratio distribution after the piezo-lens at 2000 Hz. Black cross indicates the designed focal point, dashed line indicates the energy enhancement zone. 51

3.7 Left panel: normalized kinetic energy distributions after the piezo-lens for different focal locations at 2000 Hz, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. Right panel: normalized kinetic energy along $y = 0$ lines for different focal locations at 2000 Hz, black crosses indicate the designed focal points. 54

3.8 Left panel: designed refractive index profiles. Right panel: normalized kinetic energy distributions after the piezo-lens for different refractive index profiles at 2000 Hz, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 55
3.9 Normalized kinetic energy distributions after the piezo-lens at different frequencies. Black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 56
3.10 The maximum energy enhancement ratios at different frequencies. 57
3.11 The maximum energy enhancement ratios at different frequencies for the new piezo-lens with smaller cells. 58
3.12 Normalized kinetic energy distributions after the piezo-lens for waves incident with $\theta = 20^\circ$. Black cross represents the designed focal point, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 59
3.13 Variation of the maximum energy enhancement ratio with the incident angle at different frequencies. 60
3.14 Influence of the incident angle on (a) the energy concentration zone and (b) the energy enhancement zone at 2000 Hz. Black cross represents the designed focal point. 61
3.15 Normalized kinetic energy distributions after the piezo-lens for waves excited by a point force located 1.2 m away from the left lens boundary, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 62
3.16 Influence of the point force distance on the maximum energy enhancement ratio at different frequencies. 62
3.17 Influence of the point force distance on (a) the energy concentration zone and (b) the energy enhancement zone at 2000 Hz, black cross represents the designed focal point. 63
3.18 Normalized kinetic energy distributions at different frequencies for waves excited by paraxial point force, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 64
3.19 Normalized kinetic energy distributions after the double piezo-lenses configuration for near field point force at different frequencies, black crosses indicate the designed focal points, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 65
3.20 Normalized kinetic energy distributions after the double piezo-lenses configurations at 2000 Hz for different focal locations, black crosses indicate the designed focal points, red points indicate the locations of the point forces, solid lines and dashed lines indicate the energy concentration zones and the energy enhancement zones, respectively. 66
4.1 (a) The harvesting system incorporating the adaptive piezo-lens and (b) the elementary unit in the piezo-lens. 72
4.2 An AC device. 73
4.3 Harvested power versus resistance and frequency. ........................... 75
4.4 (a) Gain ratio of harvested power versus frequency and resistance; (b) gain ratio of harvested power versus frequency when $R = 1000 \, \Omega$. 76
4.5 Energy concentration zones at different frequencies. The cross indicates the designed focal point. ................................. 77
4.6 Harvesting at different positions, the point indicates the designed focal point. ................................................................. 78
4.7 Comparison of (a) harvested power and (b) gain ratio between different cases when $R = 1000 \, \Omega$. ................................. 79
4.8 Adjusting the location of the energy concentration zone when waves are incident from the left bottom direction with an angle equal to $20^\circ$. Circles with solid line represent the energy concentration zones; crosses indicate the originally designed focal points. ............... 80
4.9 Gain ratios of harvested power at different frequencies with the original ($\alpha = 0.6/\pi$, $\beta = 0$) and adjusted ($\alpha = 0.6/\pi$, $\beta = -0.1$) piezo-lenses. Waves are incident from the left bottom direction with an angle equal to $20^\circ$ and $R = 1000 \, \Omega$. ................................... 80
4.10 The harvesting system with piezo-lens. ........................................ 81
4.11 Tested harvesters. ...................................................................... 83
4.12 The applied tone burst excitation. .............................................. 85
4.13 Performances of the harvesting system with piezo-lens when different devices are used, $Q_I = 3$. ................................. 86
4.14 Typical waveforms and converted energy when DC and SSHI-based devices are used to harvest energy from transient waves. .................. 88
4.15 Focusing transient waves. ............................................................. 89
4.16 Comparison of the performances between the cases with and without piezo-lens. ———: with piezo-lens; ——: without lens. $Q_I = 3$ .... 90
4.17 Comparison of the input power when different devices are used to harvest the transient waves, the storage capacitance for each device is optimal, the reference input power refers to the case without any harvester. ................................................................. 91
4.18 Comparison of the consumed and harvested energy in the harvesting system, the storage capacitance for each device is optimal, $Q_I = 3$. 91
4.19 Waveforms of transverse displacement and voltages when DC and SSHI-based devices are used to harvest energy from transient waves. The measure location of the displacement is the left-bottom corner on the upper surface of the piezoelectric patch for harvesting. .... 93
4.20 Comparison of the mean harvested power between the cases without and with piezo-lens. ................................................... 94
5.1 (a) A slender modulated beam lying along the $x$ axis. (b) The wave-like time-space modulation of the Young’s modulus defined by $E(x,t) = E_0 + E_m \cos(\omega_m t - k_m x)$. The time period is $T_m = 2\pi/\omega_m$, the wavelength (space period) is $\lambda_m = 2\pi/k_m$ and the wave speed is $v_m = \omega_m/k_m$. 98

5.2 Band diagrams of longitudinal modes in beams: (a) uniform beams with $\alpha_m = \beta_m = 0$, (b) space-only periodic beams with $\alpha_m = 0.4$ and $\beta_m = 0$, (c) time-space periodic beams with $\alpha_m = 0.4$ and $\beta_m = 0.2$, (d) time-space periodic beams with $\alpha_m = 0.4$ and $\beta_m = -0.2$. Shadows indicate stop bands. 101

5.3 Influences of the modulation velocity $\beta_m$ on the dispersion curves. 102

5.4 Band diagram of the beam with $\alpha_m = 0.4$, $\beta_m = 0.2$. $\Omega_m$ is the dimensionless modulation frequency, $u^+_n$ and $u^-_n$ respectively indicate the $n^{th}$ positive-going and negative-going longitudinal Bloch modes, the frequency ranges where the real part of $\mu$ is constant are stop bands of corresponding Bloch modes. 104

5.5 Amplitudes and phase velocities of harmonic components of the $u^+_0$ Bloch mode vs frequency. (a), (b): in a space-only periodic beam; (c), (d): in a time-space periodic beam. Shadows indicate stop bands. 105

5.6 Reflection at the free end of a semi-infinite time-space modulated beam. 106

5.7 Reflection and transmission at the interface between a homogeneous and a time-space modulated beam. 107

5.8 Case 1: harmonic amplitudes of the (a) incident and (b) reflected waves at $x_1 = -10\lambda_m$ when the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = 0.2$, propagating in the positive direction. Case 2: harmonic amplitudes of the (c) incident and (d) reflected waves at $x_1 = -10\lambda_m$ when the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = -0.2$, propagating in the negative direction. Amplitudes in both cases are normalized by the corresponding amplitude $|U^+_0 e^{-ik_0^+ x_1}|$ of the $0^{th}$ harmonic of the incident wave. Frequency of the $q^{th}$ harmonic is $\Omega + q\beta_m$. 110

5.9 (a), (b): Components of the (a) incident and (b) reflected waves at $\Omega_0 = 0.49$. (c), (d): Components of the (c) incident and (d) reflected waves at $\Omega_1 = 0.384$. The $(n,q)$ pixel represents the $q^{th}$ harmonic of the $n^{th}$ mode composing the wave, and the color of it indicates the corresponding amplitude normalized by the amplitude $|U^+_0 e^{-ik_0^+ x_1}|$ of the $0^{th}$ harmonic of the incident wave. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. 111
5.10 Harmonic amplitudes of the reflected waves at the interface between a homogeneous beam and a time-space modulated beam. (a): the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = 0.2$, propagating in the positive direction; (b): the modulation wave have parameters $\alpha_m = 0.4$, $\beta_m = -0.2$, propagating in the negative direction. All amplitudes are normalized by the amplitude of the incident harmonic. Frequency of the $q^{th}$ harmonic is $\Omega + q\beta_m$. 113

5.11 (a), (b): Components of the (a) reflected waves and (b) induced waves in the modulated beam at $\Omega_0 = 0.49$. (c), (d): Components of the (c) reflected waves and (d) induced waves in the modulated beam at $\Omega_2 = 0.584$. The $(n, q)$ pixel represents the $q^{th}$ harmonic of the $n^{th}$ mode composing the wave, and the color of it indicates the corresponding amplitude normalized by the amplitude of the incident harmonic. + and - signs indicate positive and negative-going harmonics, respectively, the red and green colors of them distinguish evanescent and propagative harmonics. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. 114

5.12 Frequency conversion at ends of time-space modulated beams. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. The length of the beam is $2L$, $L = 100\lambda_m$. Both the two ends of the beam are free. Arrows indicate wave propagation directions. 115

5.13 Frequency conversion at interfaces between homogeneous and time-space modulated beams. The left part of the beam ($-L \leq x < 0$) has uniform materials with $\alpha_m = \beta_m = 0$, the right part ($0 \leq x \leq L$) is a time-space modulated structure with $\alpha_m = 0.4$, $\beta_m = 0.2$, $L = 100\lambda_m$. Both the two ends of the beam are free. Arrows indicate wave propagation directions. 116

6.1 Scattering of incident waves by an 1D scatterer. 121

6.2 Scattering of incident waves by a modulated beam. 122

6.3 The beam model used in the FE simulations. The left and right parts are uniform beams, the central part is a modulated beam. Absorption Boundary Conditions (ABC) are applied at the two ends. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The lengths are $L_u = 10\lambda_m$ and $L_m = 20\lambda_m$. $O_L$ and $O_R$ are the two observation points, which are the centers of the two uniform beams, respectively. 126

6.4 Comparison between the results obtained by the FE method and the theoretical method when the modulated beam is stimulated by a left incident harmonic at $\Omega_e = 0.584$. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The dimensionless modulation frequency is $\Omega_m = 0.2$. The length of the modulated beam is $L_m = 20\lambda_m$. 127
6.5 Comparison between the results obtained by the FE method and the theoretical method when the modulated beam is stimulated by a left incident harmonic at $\Omega_e = 0.49$. The modulation parameters of the modulated beam are $\alpha_m = 0.4$, $\beta_m = 0.2$. The dimensionless modulation frequency is $\Omega_m = 0.2$. The length of the modulated beam is $L_m = 20\lambda_m$. . . . . . . . . . . . . . . . . . . . . . . . . . . . 128

6.6 Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. . . 130

6.7 Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 40\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. . . 131

6.8 Components of the induced positive-going wave group $u^+_m$ and negative-going wave group $u^-_m$ inside the modulated beam stimulated by: (a), (b) a left incident harmonic at $\Omega_e = 0.584$; (c), (d) a right incident harmonic at $\Omega_e = 0.584$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. The length of the modulated beam is $L_m = 20\lambda_m$.132

6.9 Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.4$. . . 133

6.10 Reflection and transmission coefficients for harmonics incident from two opposite directions when the length of the modulated beam is $L_m = 20\lambda_m$. The modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.6$. . . 134

6.11 Time history of the input, reflected and transmitted energy when a harmonic is incident on a space-only periodic beam within the stop band. The length of the beam is $L_m = 20\lambda_m$. . . . . . . . . . . . . . 135

6.12 Time history of the input, reflected and transmitted energy when a harmonic is incident on a time-space modulated beam within the stop band of $u^+_0$ mode. The length of the beam is $L_m = 20\lambda_m$, the modulation parameters are $\alpha_m = 0.4$, $\beta_m = 0.2$. . . . . . . . . . . . . . 135

6.13 Mean power values Vs frequency. All the results in each figure at each frequency are normalized by the corresponding incident mean power. The length of the modulated beam is $L_m = 20\lambda_m$ in all simulations. Results in the left panel are corresponding to left incident harmonics; results in the right panel are corresponding to right incident harmonics. The modulation parameters are: (a), (b) $\alpha_m = 0.4$, $\beta_m = 0.2$; (c), (d) $\alpha_m = 0.4$, $\beta_m = 0.4$; (e), (f) $\alpha_m = 0.4$, $\beta_m = 0.6$. . . . . . . . 137

6.14 Waves measured at points $O_L$ and $O_R$ in Figure 6.3 when the system is excited by a tone burst force applied at $x = 0$. . . . . . . . . . . . . . . . . . . . . . . . . . . . 138
6.15 The incident, reflected and transmitted energy when a tone burst excitation is applied at the left or right side of the modulated beam. Results are normalized by the corresponding incident energy in each simulation. The central frequency $\Omega_e$ of the excitation in each simulation is illustrated in the figures. The length of the modulated beam is $L_m = 20\lambda_m$ in all the simulations. 

6.16 A finite system contains three parts. The left and right parts are uniform beams, the central part is a modulated beam.

6.17 Spectra of waves in the system at different instants. The part between the two vertical dashed lines are the modulated beam. Arrows indicate the wave propagation directions.

6.18 Frequency responses at point $O$ shown in Figure 6.16: (a) the central part is only space periodic with $\alpha_m = 0.4$, $\beta_m = 0$; (b) the central part is a modulated beam with $\alpha_m = 0.4$, $\beta_m = 0.2$.

B.1 A generic piezo-mechanical system.

B.2 FE model of the cell containing shunted piezoelectric patches. The lower patch and the shunts are not shown in the figure.

C.1 Ideal lens on plate

C.2 Normalized power flows at 2000 Hz when plane waves are incident into the ideal lens in the $x$ direction. Black cross indicates the designed focal point.

C.3 Normalized kinetic energy distribution at 6000 Hz when plane waves are incident into the ideal lens with an angle equal to $20^\circ$.

D.1 (a) The simulation mode and (b) the applied tone burst excitation.

D.2 Comparison of the mechanical and electrical responses of the left-bottom corner on the upper surface of the piezoelectric patch for harvesting. CRM: Corrected Reduced Model, ORM: Original Reduced Model, FM: Full Model.
List of Tables

3.1 Geometry parameters of one unit piezoelectric cell in the new piezo-lens  58
4.1 Geometry parameters  ......................................................  74
A.1 Material parameters of PZ26  .............................................  147
List of publications

International Journals

• K. Yi, M. Collet, S. Karkar, Reflection and transmission of waves incident on time-space modulated metamaterials, in preparation.


International Conferences


Bibliography


[31] MA Mironov. Propagation of a flexural wave in a plate whose thickness decreases smoothly to zero in a finite interval, 1988. (Cited on pages 6, 7 and 167.)


