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Numerical simulation of axisymmetric turbulence

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Abstract

Axisymmetric turbulence is investigated using direct numerical simulations. A fully spectral method is implemented using Chandrasekhar-Kendall eigenfunctions of the curl-operator. The numerical domain is a periodic cylinder with no-penetration and partial slip conditions at the wall.

Numerical simulations are first carried out for freely decaying axisymmetric turbulence, starting from a variety of initial conditions. The simulations indicate that the global angular momentum is the most robust invariant of the system. It is further observed that large-scale coherent structures emerge, as in 2D isotropic turbulence. Energy decays more slowly than helicity, and the toroidal kinetic energy decays faster than its poloidal part. In the case where the toroidal kinetic energy becomes negligible, a quasi-two dimensional turbulence in the poloidal plane is obtained, with a behavior compatible with predictions of statistical mechanics theories.

Forced and decaying simulations are then carried out to assess the cascade-behavior of the different invariants. The existence of an inverse cascade is shown to explain the robustness of the angular momentum and the possible ‘spontaneous generation’ of this quantity and of circulation in the flow. In helical flows, the existence of a dual cascade is confirmed, with a scenario compatible with the existence of an inverse energy cascade towards the large scales, and a direct cascade of helicity towards the small scales. The inverse energy cascade seems to be mainly associated with the poloidal velocity field. Using a helical decomposition of the flow, it is shown that the direct cascade of helicity seems to subsist even in the absence of net helicity, when the ‘cascade’ of the helicity contained in oppositely polarized modes is considered individually. The scaling of the energy spectra associated with the energy cascade is compatible with elementary dimensional arguments, whereas the scaling of the inverse (presumably helicity) cascade yields an anomalously steep slope. It is shown that this slope adjusts to the value predicted by dimensional analysis when the spectra are computed from a filtered velocity field in which strong intermittent regions of velocity are not accounted for.

Finally, a preliminary (but unfortunately unfruitful) attempt is presented to apply a variational principle to the description of turbulent scalar mixing in three-dimensional turbulence.
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Nomenclature

Roman Symbols

$I_m, I_G$ Casimir invariants

$x, x_i, y_i$ Spatial coordinates

$r, \theta, z$ Cylindrical coordinates

$r_{ij}$ Distance between point vortex $i$ and $j$

$k_{eff}, k$ Effective wavenumbers

$k_n$ Axial wavenumbers

$A_{nmq}, A_{nq}$ Expansional eigenfunction

$E(k), E^{\pm}(k)$ Total energy spectrum and energy spectrum of different polarities

$E_t(k), E_p(k)$ Toroidal and poloidal energy spectrum

$E(n, q)$ Modal energy

$H_0(n, q)$ Modal circulation

$I_1(n, q)$ Modal angular momentum

$Z$ Potential enstrophy

$S$ Entropy

$E^{tot}, E$ Total energy

$E^{\pm,tot}$ Energy in polarities

$E_P$ Poloidal energy
Nomenclature

$E_T$ Toroidal energy

$H_n, H_F$ Generalized helicities

$H_0^{\text{tot}}, H_0$ Total circulation

$H_1(k), H_1^{\pm}(k)$ Total helicity spectrum and helicity spectrum of different polarities

$H_1(n, q)$ Modal helicity

$H_1^{\text{tot}}, H_1$ Total helicity

$H_1^{\pm, \text{tot}}$ Helicity of different polarities

$H$ Hamiltonian function

$L$ Length of cylindrical domain

$I_1^{\text{tot}}, I_1$ Total angular momentum

$l$ Integral lengthscale

$k_f$ Forcing wavenumber

$B$ Magnetic field

$I_{nmq}, I_{nq}$ Normalisation coefficient

$R$ Radius of cylindrical domain

$\text{Re}$ Reynolds number

$k_E, k_\Omega, k_l, k_H$ Dissipative scales

$T_X(k)$ Transfer spectrum of quantity $X$

$T_X(n, q)$ Modal transfer function

$u_r, u_\theta, u_z$ Velocity components in cylindrical coordinates

$e_r, e_\theta, e_z$ Unit vector in cylindrical domain

$y$ Radius related coordinate in axisymmetric turbulence, $y = \frac{r^2}{\tau}$

Greek Symbols

$\sigma$ Local angular momentum
<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>ξ</td>
<td>Potential vorticity $\xi = \frac{\omega}{r}$</td>
</tr>
<tr>
<td>Λ</td>
<td>Control parameter of bifurcation equilibria in 2D turbulence or axisymmetric turbulence</td>
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<tr>
<td>ρ</td>
<td>Density</td>
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<tr>
<td>$\tau_{xy}$</td>
<td>Aspect ratio</td>
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<tr>
<td>Ω</td>
<td>Enstrophy</td>
</tr>
<tr>
<td>Ωₚ</td>
<td>Toroidal enstrophy</td>
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<tr>
<td>Ωₜ</td>
<td>Poloidal enstrophy</td>
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<tr>
<td>$\xi_v, \pm \lambda_{nmq}$</td>
<td>Expansional coefficient of polarities</td>
</tr>
<tr>
<td>$\xi_v, \xi_v, \xi_v, \xi_v$</td>
<td>Expansional coefficient</td>
</tr>
<tr>
<td>$\alpha(n, q)$</td>
<td>Forcing amplitudes</td>
</tr>
<tr>
<td>$\alpha^{\pm \lambda}(n, q)$</td>
<td>Forcing amplitudes of polarities</td>
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<tr>
<td>$\Gamma_n$</td>
<td>Casimirs of the potential vorticity</td>
</tr>
<tr>
<td>$\alpha_n, \beta, \zeta(r)$</td>
<td>Lagrange multipliers</td>
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<tr>
<td>Δ</td>
<td>Laplacian</td>
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<tr>
<td>$\lambda_{nmq}, \gamma_{nmq}, \lambda_{nq}, \gamma_{nq}$</td>
<td>Parameters in numerical method</td>
</tr>
<tr>
<td>$\rho(r)$</td>
<td>Probability distribution of pure mixing quantity at r</td>
</tr>
<tr>
<td>$\rho_{\pm}(r)$</td>
<td>Probability distribution of + and − point vortices</td>
</tr>
<tr>
<td>Δₚ</td>
<td>Pseudo-Laplacian</td>
</tr>
<tr>
<td>$\kappa_i$</td>
<td>Point circulations</td>
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<tr>
<td>$\psi$</td>
<td>Stream function</td>
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<tr>
<td>$\Pi_X(k)$</td>
<td>Flux rate spectrum of quantity X</td>
</tr>
<tr>
<td>τ</td>
<td>Turn over time</td>
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<tr>
<td>ν</td>
<td>Kinematic viscosity</td>
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\( \omega \)  Vorticity

\( \omega_r, \omega_\theta, \omega_z \)  Vorticity components
Chapter 1

Introduction, outline and objectives

1.1 How to describe turbulence

Turbulence in nature and engineering applications is characterized by its irregular, chaotic and difficultly predictable flow patterns. Many theoretical, experimental and numerical efforts have been made to better understand the physical principles which determine this dynamics. A difficulty in finding a detailed description is that the number of degrees of freedom of a turbulent flow is a rapidly increasing function of the Reynolds number. A precise description of high Reynolds number turbulent flows needs therefore millions or even billions of modes [1], in order to discretize all the scales. As in molecular dynamics, where it is not the precise knowledge of positions and velocity of all atoms, but only their collective average, which is of interest, in the study of turbulent flows it is also the statistical averages only which are of interest in most applications. It is therefore tempting to apply the ideas of statistical mechanics, developed for molecular dynamics, to turbulent flows. However, statistical mechanics is most easily applied to thermodynamic equilibrium or quasi-equilibrium situations, and classical turbulence is an intrinsic non-equilibrium problem.

1.2 Statistical mechanics and turbulence

1.2.1 Two-dimensional turbulence

In Chapter 2 we review the development of a statistical mechanics framework for turbulence. The successful development of a statistical mechanics approach of turbulent flows started with the work of Onsager [2, 3] on the description of two-dimensional turbulence using point-vortices. Subsequent theoretical developments of Joyce and Montgomery [4–6] and Lundgren & Pointin [7] formalized Onsager's ideas (see also the review by Kraichnan and Montgomery
Introduction, outline and objectives

(8)). It was shown in simulations of decaying two-dimensional Navier-Stokes turbulence that the theoretical predictions of the point vortex model were surprisingly accurately verified. Indeed, the presence of viscous dissipation did not seem to invalidate the statistical mechanics approach derived from a discrete inviscid system.

To go beyond the point-vortex case, in order to apply statistical mechanics to the continuum case, Miller [9] and Robert and Sommeria [10] introduced a coarse-grained description of the vorticity field. Introducing a mixing entropy and optimizing the latter under constraints of conserved energy and other invariants, they managed to extend the statistical mechanics approach to the case of the two-dimensional Euler equations. Subsequently, using statistical mechanics, explanations were given for the behavior of the Great Red Spot on Jupiter [11, 12], the cyclones and large-scale ocean currents on Earth [13–15]. In all these situations, the inertial terms dominate, and the dynamics can be considered governed by the incompressible Euler system.

Obviously one would like to apply these ideas to the more general case of three-dimensional turbulence. However, the dissipative anomaly, i.e., a non-vanishing energy dissipation in the limit of vanishing viscosity (see for instance [16, 17] for experimental and numerical evidence) distinguishes drastically the physics of three-dimensional turbulence from the two-dimensional case. However, specific cases of three-dimensional flows, where the large-scale structure is closer to the two-dimensional case, might be amenable to treatment by statistical mechanics.

1.2.2 Axisymmetric turbulence

This possibility motivated a number of studies on axisymmetric turbulence, i.e., axisymmetric Euler flows in cylindrical geometry. Indeed, the axisymmetric Euler (or Navier-Stokes) equations consist of three, coupled velocity components, varying along two spatial directions. This situation is therefore somewhere between two- and three dimensional flows. The interest in such flows is mainly academic, since it is very difficult to generate a real turbulent flow which will remain invariant along the azimuthal direction. An exception might be turbulence in the presence of a strong azimuthal magnetic field, but these kind of flows, encountered in magnetically confined nuclear fusion devices, such as tokamaks have other experimental difficulties to cope with. Only on average experimental flows are sometimes axisymmetric. This is the case of the Von Kármán flow, a cylindrical flow between rotating plates, or impellers [18] and we will come back to this.

Similar to 2D turbulence, axisymmetric inviscid turbulence also possesses an infinite number of conserved quantities, such as the total energy, the moments of the angular momentum and generalized helicities. Pioneering theoretical works [19, 20] suggested that the tools
developed for 2D flows can be transposed to axisymmetric flow, and lead to a new series of variational principles. Leprovost and collaborators [20] characterized the general stationary solutions of the axisymmetric Euler equations. Later on, experiments in a Von Kármán flow by Ravelet et al. [21] confirmed the formation of large-scale structures predicted by these theories. Monchaux et al. [22] also observed that at large values of the Reynolds number, the stationary states tend to select the shape of a Beltrami-flow pattern [23], which means a dominant alignment of velocity and vorticity in the flow. Naso et al. [24, 25] further developed the ideas of Leprovost et al. And in the context of a Taylor-Couette geometry, Thalabard et al. [26] defined yet another theoretical framework to predict the statistical mechanics of the axisymmetric Euler flows in a Taylor-Couette geometry. Indeed, as mentioned above, some experimental support was (perhaps surprisingly) obtained for the theoretical developments [22], by considering the time-averaged component of the Von Kármán flow.

However, in practice the theory was developed for the purely axisymmetric case and this case can only be precisely reproduced in numerical simulations. To the best of our knowledge, such simulations have never been carried out and the present investigation aims at giving the first results using direct numerical simulations of purely axisymmetric turbulence.

1.3 Numerical method

The numerical method we use to study strictly axisymmetric turbulence is discussed in Chapter 3. We use a fully spectral numerical method. We modify and further develop an existing code for flow in a cylindrical geometry. The method is based on an expansion of the velocity field using Chandrasekhar-Kendall helical eigenfunctions of the curl [27]. These functions form a complete and orthogonal basis in cylindrical geometry [28–30].

This type of method has been applied to the numerical study of 2D turbulence in a circular geometry [31–33] and 3D Magnetohydrodynamics in cylinder geometry [34–37]. We will keep the radial non-penetration and axially periodic boundary conditions of the original code, since those are the most natural conditions using our method, and since they correspond to a physically interesting case, where the axial motion is not constrained by solid boundaries.

Our study is the first numerical investigation of axisymmetric turbulence and must be considered of exploratory nature. The results of the simulations are reported in Chapters 4 and 5.
1.4 Freely decaying axisymmetric turbulence

Using our numerical method, we investigate in Chapter 4 the self-organization properties of axisymmetric turbulence, starting from different sets of initial conditions. One of the questions that we want to address is whether the flow self-organizes into coherent flow structures, and if these structures have a shape which is predictable by statistical mechanics. In connection with this, we want to know which global quantities are most robust under the combined action of nonlinear transfer and viscous dissipation.

We consider flows with and without average angular momentum, one of the key invariants of the theoretical contributions [20, 24], which was also shown to play a major role in freely decaying 2D turbulence in the presence of no-slip rigid circular walls [31].

A distinct dynamics of the toroidal and poloidal flows will be illustrated for some cases, where it will be shown that poloidal energy is more robust than toroidal energy. Also, in some cases, the flows tend to a quasi 2D dynamics [19], reproducing features of the Miller, Robert, Sommeria theory.

Why certain invariants are more robust than others is related to their cascade properties, a topic which is addressed in the following.

1.5 Spectral dynamics

In Chapter 5, the cascade properties of the different invariants is investigated, in particular of energy and helicity, by considering the spectra and fluxes of energy and other invariants.

Indeed, in the context of the Von Kármán experiment [38], energy spectra were measured for a wide range of Reynolds numbers and cascade directions of the mean energy and helicity were inferred from them. A dual energy cascade seemed to be observed, with a direct helicity transfer and an inverse energy transfer. They proposed phenomenologically the exponents of the energy spectra assuming these. We compare our simulation results with the experimental ones. For this, we implement an artificial force term by adding a negative viscosity to modes in a fixed wavenumber band, in order to keep the flow in a statistically stationary state.

The cascade properties of 2D turbulence have been thoroughly investigated by many researchers (e.g. [39–41]). The joint cascade of energy and helicity has also been profoundly studied in the 3D isotropic turbulence [42–44]. It is therefore fairly interesting to investigate these mechanisms in the axisymmetric case, to check whether the characteristics are compatible with a case intermediate between 2D and 3D. The statistically stationary simulation, may allow us to look into this issue, even at the moderate resolutions we consider.
1.6 An attempt to describe three-dimensional mixing using variational methods

The use of statistical mechanics to describe 2D Euler turbulence and 3D axisymmetric Euler turbulence does provide important new point of view for understanding the dynamics of turbulence in general. It would be interesting to see if statistical mechanics could be used to explain phenomena in more general 3D turbulence.

In the 3D Navier-Stokes equation, the non-linear term is responsible for the chaotic behavior of the turbulence. Kraichnan and Panda [45] observed that the variance of this nonlinear term is smaller in a turbulent velocity field than in its Gaussian counterpart (a flow with the same energy spectrum but random phases), and called this phenomenon depression of nonlinearity. One of the geometrical explanations of such a reduced nonlinearity is the Beltramization, the tendency of alignment of velocity and vorticity. However Kraichnan and Panda proposed that there may be a more general mechanics behind the depression. Recently, Bos et al. [46] have also observed similar statistical reduction for the variance of the advection term of the passive scalar. This observation supports the proposition of Kraichnan and Panda and indicates that the mechanism is more general.

In Appendix A, we check a conjecture that the depletion of nonlinearity or advection corresponds to a statistically optimized state, where the nonlinearity is minimized under constraints. We check a number of different constraints to verify this conjecture in three-dimensional turbulent passive scalar mixing. These investigations are of very exploratory nature and are not based on solid (theoretical or numerical) grounds, and that is why they are presented in an appendix.

1.7 Objectives

In summary, the main objectives of this thesis are to:

- Implement a spectral method based on the eigenfunction of the curl for axisymmetric turbulence in a cylindrical geometry.

- Simulate and analyze a variety of physical properties of freely decaying axisymmetric turbulence; check theoretical predictions.

- Study the cascade properties in these decaying flows and in forced stationary axisymmetric turbulence.
• Propose and explore (as a very preliminary approach) a possible principle of minimization for 3D turbulent mixing.
Chapter 2

Theoretical Background

The main motivation to investigate axisymmetric turbulence is its conceptual position somewhere between two- and three-dimensional turbulence. Indeed, axisymmetric turbulence is fully described by three velocity components (as in 3D), which vary only along two coordinates (as in 2D). Since for 2D flows statistical mechanics has been successfully applied to predict and quantify its tendency to develop large-scale structures, something which has not been achieved in three dimensions, the intermediate case is now in the center of attention to help us go beyond the 2D case. We will outline the developments that have led to the formulation of a theory. In section 2.1 we will discuss the statistical mechanics of two-dimensional turbulence, in section 2.2 the axisymmetric case, and in section 2.3 we introduce some notions on the dynamics of these flows, more specifically focusing on turbulent cascades.

2.1 2D turbulence

2.1.1 The 2D Navier-Stokes equation

The dynamics of an incompressible fluid in two-dimensions is described by the Navier-Stokes equations,

\[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} = 0, \]

where \( \mathbf{u} = (u, v, 0) \) is the velocity, \( p \) the pressure divided by density, and \( \nu \) the kinematic viscosity. The introduction of the vorticity \( \omega = \nabla \times \mathbf{u} \), allows to simplify this set of equations. In two dimensions only one component of the vorticity is nonzero, \( \omega = (0, 0, \omega) \). Taking the
curl of equation (2.1), the equation for this component of the vorticity becomes,

\[ \partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega. \]  

(2.3)

Since the vorticity can be computed from the velocity, this equation describes the same dynamics as equations (2.1). Introducing the stream-function \( \psi \), such that

\[ \omega = -\Delta \psi \]  

(2.4)

and,

\[ u = -e_z \times \nabla \psi = (\partial_y \psi, -\partial_x \psi) \]  

(2.5)

allows us to write a single equation, containing only the stream-function,

\[ \partial_t \Delta \psi + \{ \Delta \psi, \psi \} = \nu \Delta^2 \psi, \]  

(2.6)

in which the Poisson brackets are defined as \( \{ a, b \} = \partial_x a \partial_y b - \partial_y a \partial_x b \). In the following we will prefer to write this equation in the equivalent form

\[ \partial_t \omega + \{ \omega, \psi \} = \nu \Delta \omega. \]  

(2.7)

An interesting property of Poisson brackets is that if there is a functional relation between \( a \) and \( b \), i.e. \( a = f(b) \), then \( \{ a, b \} = 0 \). Indeed, we find,

\[ \{ f(b), b \} = \partial_b f \partial_y b \partial_x b - \partial_y f \partial_x b \partial_y b = 0. \]  

(2.8)

The term \( \{ \omega, \psi \} \) in equation (2.7) represents the nonlinearity of the Navier-Stokes equation, physically corresponding to the advection term. In the absence of this term, the dynamics become linear and purely viscous. Therefore, if in a flow-field the vorticity is a function of the stream-function \( \omega = f(\psi) \), it means that we have a flow depleted from nonlinearity. In the inviscid case, i.e., \( \nu = 0 \), the resulting equation \( \partial_t \omega = 0 \) shows that we have then a steady (or stationary) solution. A vorticity field that does not evolve in time is by definition very coherent. A close-to-functional relation \( \omega = f(\psi) \) can thus indicate that the corresponding flow-field is very coherent. We will show in the following that there is indeed a relation between two-dimensional long-living structures and the observation of a relation \( \omega = f(\psi) \).
2.1 2D turbulence

2.1.2 Invariants of the inviscid equations

To simplify the considerations, we will consider the spatially homogeneous case, or, for numerical convenience, space-periodic case. The evolution of the kinetic energy of the Navier-Stokes equations can be written for this case, in the absence of an external forcing mechanism, as

$$\frac{1}{2} \partial_t \langle u_i u_i \rangle = -\nu \langle \partial_j u_i \partial_j u_i \rangle,$$

(2.9)
in which $\langle \rangle$ means the volume integration and it shows that in the absence of viscosity, the kinetic energy is conserved (Note that the limit of small $\nu$ is different from the case $\nu = 0$). The kinetic energy is therefore an invariant of the inviscid system, independent of the spatial dimension. The kinetic energy is the only universal invariant of the Navier-Stokes equations. Other invariants exist, specific for a particular dimension or symmetry. For instance in three dimensions, the helicity $\langle u_i \omega_i \rangle$ is also conserved in the absence of viscosity. In two-dimensions, equation (2.9) can be rewritten (for bounded or periodic domains) as

$$\frac{1}{2} \partial_t \langle \omega \psi \rangle = -\nu \langle \omega^2 \rangle.$$

(2.10)

Similar equations can be derived for the evolution of $\langle \omega^n \rangle$ for all integer values of $n > 0$. All these quantities are therefore invariants of the system and will play an important role in the following. In particular the circulation $\langle \omega^1 \rangle$ and enstrophy $\langle \omega^2 \rangle$ will be important in the following. Note that in the specific case of a circular two-dimensional domain, another invariant, the angular momentum, defined as $\langle \psi \rangle$ appears.

2.1.3 The conjecture of Onsager

An original idea which somehow motivated all later applications of statistical mechanics was due to Onsager [2]. In 1949, he proposed that the formation of self-organized structures can be explained as a consequence of equilibrium statistical mechanics at negative absolute temperature. Indeed, the temperature of a system is related to the energy $E$ and the entropy $S$ by the relation

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$ 

(2.11)

In classical systems, where the temperature is positive, this relation shows that when energy is injected in a system, the entropy will increase. The point-vortex system that we will now discuss is different.

Onsager considered a Hamiltonian system of $n$ parallel point-vortices of intensities $\kappa_1, \ldots, \kappa_n$ (circulations) representing an incompressible Euler flow. The equations of motion
of these vortices are the Hamiltonian expressions

\begin{align}
\kappa_i \frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i}, \\
\kappa_i \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i},
\end{align}

in which \( t \) denotes the time, \( x_i, y_i \) represent the coordinates of the vortex \( i \) (in this section indexes do not represent coordinate directions as in the last section) and \( H \) is the Hamiltonian of the form

\begin{equation}
H = -\frac{1}{2\pi} \sum_{i>j} \kappa_i \kappa_j \log \left( \frac{r_{ij}}{L} \right),
\end{equation}

\begin{equation}
r_{ij}^2 = (x_j - x_i)^2 + (y_j - y_i)^2.
\end{equation}

The variables \( x_i \) and \( y_i \) are called the conjugate variables of the system and \( L \) is an interaction length. It is a particular feature of the point-vortex system that it is the two spatial coordinates only which are the conjugate variables. In standard molecular dynamics of an ideal gas the conjugate variables of the system are the positions and velocities of the molecules. The phase-space of the system is then infinite in the sense that the number of possible states is unbounded, even in a bounded volume, since the velocity is not bounded (in a non-relativistic description). In other words, an infinite number of possible states is possible for a given global energy, if no constraints are given on the maximum of the individual molecular velocities.

On the contrary, in the point vortex system, the velocity is not a conjugate variable, only the positions of the vortices are. When the vortices are confined in a finite area and ergodicity is assumed to hold, Onsager pointed out that the phase-space of this system is finite. Indeed, the volume of the system will limit the different positions that the vortices can take.

The consequence of this observation is that when kinetic energy is fed into the system, the entropy cannot grow indefinitely. Above a given threshold value of the energy (related to the density of vortices of a certain strength), the only way for the system to stock more energy is to approach vortices of equal sign. Onsager showed that for the simple case of two vortices the energy increases when they are approached, and conjectured that for a set of vortices this would remain true. Re-arranging (and thereby ordering) a random spatial distribution, corresponds to decreasing the entropy. From equation (2.11) it is immediately clear that a decreasing entropy for an increasing energy corresponds to a negative temperature.

The contribution of Onsager was a mix of very clever ideas and strong intuition. The conjecture was confirmed later by more rigorous approaches that we will discuss now. For a historical overview of the contributions of Onsager and the later works, we refer to reference [3].
2.1.4 Development of the ideas by Joyce & Montgomery

The system described above of interacting point-vortices is equivalent to the system which describes the interaction of charged rods aligned with a uniform magnetic field. Joyce and Montgomery [5, 6], studied this system, also called the 2D electrostatic guiding center plasma. They managed to derive an equilibrium state for the vorticity distribution. We will develop their derivation in some detail here, since it is illustrative for all the further, slightly more complicated ideas.

Joyce and Montgomery considered a field containing $N$ point vortices (or charged rods) of positive charge $1/N$ and $N$ point vortices of negative charge $-1/N$, resulting in a globally charge neutral plasma. For the case of point-vortices, this means that the global circulation is zero and that the circulations are constant and equal for all vortices of the same sign. The Hamiltonian system is then described by expression (2.12) where all values of $\kappa$ are equal to either $-1/N$ or $1/N$. In this case, comparing (2.12) and (2.5), it is clear that the Hamiltonian $H$ is very similar to the stream-function $\psi$ in two-dimensions. The difference is that the stream-function describes a continuous field, whereas the Hamiltonian describes a finite number of point-vortices. If one wants to describe the vorticity field as a function of the spatial position, the point-vortex model corresponds to a distribution of the vortex densities given by the expressions

$$\rho_\pm(r) = \frac{1}{N} \sum_{i=1}^{N} \delta(r - r_\pm^i).$$

(2.15)

This distribution corresponds to the vorticity density field,

$$\omega(r) = \rho_+(r) - \rho_-(r).$$

(2.16)

The vorticity field is related to a stream-function, $\omega = -\Delta \psi$, where the stream-function is given by the Hamiltonian (2.14). The total kinetic energy of the system is given by

$$E = \frac{1}{2} \int \psi \omega d^2r.$$

(2.17)

From expression (2.15), an entropy was defined

$$S = - \int d^2r \rho_+(r) \log \rho_+(r) - \int d^2r \rho_-(r) \log \rho_-(r).$$

(2.18)

In this system, as in Onsager’s analysis, energy $E$ is conserved and the number of vortices $N$ is fixed. Since the number of vortices is fixed, the density of vortices integrated over space is
constant and is equal to the global circulation

\[ \Gamma = \int (\rho_+(r) - \rho_-(r))d^2r. \]  
(2.19)

Applying equilibrium statistical mechanics corresponds to maximizing the entropy for given values of \( E \) and \( \Gamma \). This can be expressed as a variational equation

\[ \delta S - \beta \delta E - \alpha \delta \Gamma = 0. \]  
(2.20)

where \( \beta \) and \( \alpha \) are Lagrange multipliers. The first one, \( \beta \) corresponds here to the inverse of Onsager’s vortex-temperature. Combining this equation with the expressions for \( E \) and \( S \), we have

\[ \delta \int [-\rho_+ \log \rho_+ - \rho_- \log \rho_- - \beta \psi (\rho_+ - \rho_-) - \alpha (\rho_+ - \rho_-)]d^2r = 0. \]  
(2.21)

Since the integrand depends only on the variable \( \rho_+ \) and \( \rho_- \), its derivatives with respect to those two quantities should be equal to zero for an optimum. Deriving the integrand with respect to \( \rho_+ \) and \( \rho_- \) yields the equations

\[ 1 + \log \rho_+ + \beta \psi + \alpha = 0 \]  
(2.22)

\[ 1 + \log \rho_- - \beta \psi - \alpha = 0 \]  
(2.23)

equivalent to

\[ \rho_+ = \exp(-(1 + \beta \psi + \alpha)) \]  
(2.24)

\[ \rho_- = \exp(-(1 - \beta \psi - \alpha)) \]  
(2.25)

Using (2.16), this yields immediately

\[ \omega = \exp(-1)(\exp(-(\beta \psi + \alpha)) - \exp(\beta \psi + \alpha)) \sim \sinh(-(\beta \psi + \alpha)). \]  
(2.26)

We find thus that the vorticity is a functional relation of the stream-function, a result which implies a steady solution of the Euler-equations, as discussed in section 2.1.1. Even though inviscid point-vortex dynamics is obviously different from Navier-Stokes turbulence, it was shown later in simulations of freely decaying 2D turbulence on a double-periodic domain [47], that the system relaxed to a system where the functional relation (2.26) is well verified.
Returning now to equation (2.7), we see that this is indeed a non-trivial state of 2D turbulence, where the nonlinearity vanishes.

The contribution of Joyce and Montgomery provided a more rigorous theory for the relaxation of the point-vortex model to an equilibrium state, and placed Onsager’s conjecture on more solid ground. However, compared to the continuum description of real fluid turbulence, it has some drawbacks. For instance, the effect of incompressibility is not taken into account. For very high densities of point-vortices, the system will therefore start to behave differently from the continuum description. Indeed, the point-vortex model and the approach of Joyce and Montgomery yields the low density limit of a continuum theory that we will now describe.

2.1.5 The Miller-Robert-Sommeria theory

The observation that the solution of simulations of 2D Navier-Stokes turbulence evolved towards a state roughly in agreement with Joyce and Montgomery’s prediction (2.26) logically encouraged to define a continuum description, to go beyond the point-vortex model. Such a continuum description was successfully proposed, almost simultaneously, by Robert & Sommeria and Miller [9, 48].

The difficulty to derive a continuum description of the Euler-equations is obviously the micro-structure of the vorticity field. Indeed, the mixing of the vorticity field through the advection term in equation (2.3), leads to finer and finer structures. The physical picture is similar to the mixing of a blob of scalar in a fluid flow, when the diffusivity of the scalar is extremely small. In the case of the mixing of the vorticity given by the Euler-equations, there is no physical reason that the thinning of the vorticity-filaments is limited. At long enough times, the thickness of the filaments will therefore always go below the minimum observable scale, or even below the scale at which the continuum description holds.

However, the goal of the theory is not the prediction of the fine-scale structure of the vorticity field, but that of the large scale structure. It is therefore possible to define a grain-size, corresponding to the size at which observations are still possible. For the smaller scales, the essential ingredient of the Robert & Sommeria and Miller theory is the introduction of a probabilistic description of the micro-structure of the vorticity field. They therefore introduced the local macroscopic vorticity probability distribution $\rho(\omega, r, t)$, which denotes the probability that the vorticity takes a value $\omega'$ in the range of $\omega \leq \omega' < \omega + d\omega$ at the position $r$ and time $t$. The distribution function is defined such that

$$\int \rho d\omega = 1.$$ (2.27)
The local average vorticity is computed as

$$\overline{\omega}(r,t) = \int \rho(\omega, r, t) \omega d\omega.$$  \hspace{1cm} (2.28)

The so-defined coarse-grained vorticity can then be understood as the smallest scale at which we observe the dynamics. This essential step is possible in 2D turbulence, since the enstrophy is a quantity which is purely mixed. Indeed, its evolution is governed by an advection-diffusion equation. In three-dimensional turbulence, vorticity could be created in the sub-grain scales, since the vorticity equation also contains a production (or vortex stretching) term, making the description more complicated.

Starting from this coarse-grained continuum approach, they defined the mixing entropy of vorticity as

$$S(t) = -\int (\rho \log \rho) d^2 \omega,$$  \hspace{1cm} (2.29)

and demonstrated that the most probable macrostate could be achieved by maximizing this entropy under the constraints of fixed (coarse grained) invariants.

The function $\rho$ can then be determined by solving the variational problem:

$$\delta S - \beta \delta \overline{E} - \sum_{n=1}^{+\infty} \alpha_n \delta \overline{\Gamma_n} = 0,$$  \hspace{1cm} (2.30)

where $\beta$ and $\alpha_n$ are Lagrange multipliers. The energy $\overline{E}$ and $\overline{\Gamma_n}$ are coarse grained invariants of 2D Euler flow [9, 48],

$$\overline{E} = \frac{1}{2} \int d^2 x \overline{\omega \psi} = \frac{1}{2} \int d^2 x d\omega \rho \omega \psi,$$ \hspace{1cm} (2.31)

$$\overline{\Gamma_n} = \int d^2 x \overline{\omega^n} = \int d^2 x d\omega \rho \omega^n.$$ \hspace{1cm} (2.32)

Therefore, expression (2.30) can also be written as

$$\int \left( \delta (\rho \log \rho) - \frac{\beta}{2} \delta (\rho \omega \psi) - \sum_n \alpha_n \delta (\rho \omega^n) \right) d^2 x d\omega = 0.$$ \hspace{1cm} (2.33)

Deriving this expression with respect to $\rho$ gives then the result,

$$\rho = \frac{1}{Z_{\text{norm}}} \exp(-\beta \omega \psi/2 - \sum_n \alpha_n \omega^n),$$ \hspace{1cm} (2.34)

where $Z_{\text{norm}}$ is a normalizing factor which ensures that (2.27) holds. The Robert & Sommeria and Miller theory allowed thus a rigorous statistical mechanics description of the continuum
case. It further allowed to account for all invariants of the 2D Euler equation as constraints in
the determination of the equilibrium states for all possible initial conditions.

### 2.1.6 Further developments, fragile constraints and selective decay

A difficulty with the theory discussed in the previous section is the fact that it depends on
an infinite number of invariants. Therefore, in order to do any prediction, all $\Gamma_n$ are to be
prescribed. In practice this is not feasible. It is therefore tempting to take into account only a
finite number, but is this justified?

An important observation is that the coarse-graining procedure affects not all quantities
in the same manner. Let us illustrate that here. As usual in the statistical description of
turbulence, any averaging or coarse-graining procedure affects the second and higher order
moments of an averaged quantities. Indeed,

$$\bar{\omega}^n \neq \bar{\omega}^n$$

for integer values of $n > 1$. Therefore, for any quantity which depends linearly on $\omega,
\overline{f(\omega)} = f(\bar{\omega})$. For these quantities, the coarse-grained invariants can be computed directed
from the coarse-grained vorticity. Examples are the energy and the circulation,

$$\overline{E(\omega)} = \overline{\int \omega \psi dx dy} = \int \bar{\omega} \psi dx dy = E(\bar{\omega}) \quad (2.36)$$
$$\overline{\Gamma_1(\omega)} = \overline{\int \omega dx dy} = \int \bar{\omega} dx dy = \Gamma_1(\bar{\omega}). \quad (2.37)$$

These quantities were therefore baptized robust constraints, since their coarse grained value is
independent of the exact micro-structure. For all other invariants, for instance the enstrophy,
this is not the case,

$$\overline{\Gamma_2(\omega)} = \overline{\int \omega^2 dx dy} \neq \overline{\Gamma_2(\bar{\omega})} = \int \bar{\omega}^2 dx dy \quad (2.38)$$

Indeed, the fine-scale structure affects the coarse grained invariants. This shows that the
invariants $E$ and $\Gamma_1$ are distinct from the other invariants, which might not be conserved
under coarse graining. It has been proposed by several contributions [12, 49–51] that the
conservation of all the constraints of the 2D Euler equation is abusive. Considering the
occasion where the dissipation and forcing that interrupting the fluctuations may leave the
invariants like $E$ and $\Gamma_1$ to be more robust, a simplified thermodynamic approach based on
Miller-Robert-Sommeria theory with only robust constraints have been studied [52, 53].
In parallel with the development of a statistical mechanics description of turbulence, phenomenological theories were also proposed to explain the observations. One of these theories was the conjecture of selective decay [39, 54, 55]. This conjecture states that the dynamics of a system are determined by the minimization of certain invariants while keeping other invariants fixed. The idea is that certain quantities are more robust than others. For instance it was proposed in the framework of two-dimensional turbulence that energy is more robust than enstrophy [39]. The variational principle for two dimensional turbulence (for the special case of zero circulation) would then become

$$\delta \Gamma_2 - \beta \delta E = 0. \tag{2.39}$$

which can be written, using the vorticity density function, as

$$\delta \int (\rho \omega^2 - \beta \rho \omega \psi) \, dxdy = 0, \tag{2.40}$$

yielding, after derivation of the integrand with respect to \(\rho\) the relation,

$$\omega^2 - \beta \omega \psi = 0, \tag{2.41}$$

and therefore

$$\omega = \beta \psi. \tag{2.42}$$

This is also a functional relation, but linear in \(\psi\). In reference [15, 53] it was shown rigorously that the selective decay principle, stated as the variational equation (2.39) is in fact equivalent to the more rigorous approach by Miller, Robert and Sommeria for a given number of invariants. It justified thereby the use of selective decay, when the statistical mechanics approach is too complicated. Since a Taylor-expansions gives

$$\sinh(x) = x + x^3/3! + x^5/5! + \ldots, \tag{2.43}$$

a linear relation for \(\omega = f(\psi)\) is in agreement with the observed \(\sinh(\psi)\) for small values of \(\omega, \psi\) (the limit of small energy). The authors state that nonlinear relations can be obtained by retaining more fragile invariants. An exact determination of the sinh function would in principle request to retain all invariants in this phenomenological description. However, if one discards the extreme values of the vorticity, a reasonable approximation should in principle be obtained by retaining only the first invariants. This approach has two advantages: firstly it shows that a finite number of invariants allows to describe to a certain approximation the system. Secondly, it avoids the introduction of an entropy in the description, rendering
Fig. 2.1 (Figure from [53]) Series of equilibria in a square domain ($\tau_{xy} = 1 < 1.12$).

Fig. 2.2 (Figure from [53]) Series of equilibria in a rectangular domain for which $\tau_{xy} = 2 > 1.12$.

the variational equation substantially simpler, and this will be helpful in the following, if more complicated systems are considered.

Using these tools, the most probable states in a closed rectangular domain where computed as a function of the control parameter $\Lambda = \Gamma / \sqrt{2E}$ [52, 53]. These results, firstly shown in Figure 2.1 for smaller aspect rate case ($\tau_{xy} < 1.12$), illustrate the power of the statistical mechanics approach: without solving the Navier-Stokes equations, it can be shown what the post probable statistical solution is. Moreover, the results for larger aspect rate case ($\tau_{xy} > 1.12$), are shown in Figure 2.2. In particular it is observed that the maximum entropy state can evolve from a dipole to a monopole by increasing the value of $\Lambda$. 
2.1.7 Summary of statistical mechanics of 2D Euler flows

Since the work of Onsager, the statistical mechanics of two-dimensional turbulence have been shown to be a powerful approach to predict the most probable states of high Reynolds numbers 2D flows both in infinite and bounded domains. Geophysical flows [56], Jupiter’s red spot [11] and oceanic flows [13–15] can all be treated using the same type of formalism, based on maximum entropy states, or selective decay principles. In the following section we will discuss the extension of these ideas to the case of axisymmetric turbulence.

2.2 Axisymmetric turbulence

At variance with 2D, applying statistical mechanics to 3D turbulence faces some essential difficulties. Classical statistical mechanics focus on equilibrium states of the physical system. In 2D turbulence the dissipation goes to zero when the Reynolds number tends to infinity. Thereby, the invariants of the Euler equation are almost conserved for high Reynolds number Navier-Stokes turbulence. In contrast, 3D Navier-Stokes turbulence is observed to have a finite dissipation even at infinitely high Reynolds number (illustrated experimentally and numerically in references [16, 17]). This complicates the application of statistical mechanics to 3D flows since the invariants of the Euler equation are not conserved for the viscous system in 3D. The system, in the absence of forcing might not have time to relax to a possible equilibrium, since the finite dissipation will rapidly dissipate the energy of the system. Furthermore, the pure mixing of vorticity, characteristic of 2D Euler flows is not satisfied for the 3D case. Even without viscosity, the stretching of the vorticity will destroy the pure mixing property.

Recently, it was proposed to study the statistical mechanics of axisymmetric turbulence [20, 24, 25]. The flow is assumed to be 3D and symmetric with respect to rotations around a fixed axis. The statistical approach of Miller-Robert-Sommeria theory was applied to this case. The results of this approach were qualitatively compared to data available for the experimental Von Kármán flow [21], which is only axisymmetric in the statistical (average) sense. The large-scale self-organized structures observed in this flow were in qualitative agreement with the theoretical results. Further contributions using an alternative approach using a long range lattice model were then established by Thalabard et al. [26]. So far the theories agree qualitatively with the experiments and provide insights into the physical mechanisms characteristic of this special case of 3D turbulence. In this section we will introduce in detail some of the important results.
2.2 Axisymmetric turbulence

2.2.1 Equations for axisymmetric Euler flows

The equations for the incompressible axisymmetric Euler equations can be written in cylindrical coordinates \((r, \theta, z)\) as

\[
\frac{1}{r} \partial_r (ru_r) + \partial_z u_z = 0, \tag{2.44}
\]

\[
\partial_t u_r + u_r \partial_r u_r + u_z \partial_z u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \partial_r p, \tag{2.45}
\]

\[
\partial_t u_\theta + u_r \partial_r u_\theta + u_z \partial_z u_\theta + \frac{u_\theta u_r}{r} = 0, \tag{2.46}
\]

\[
\partial_t u_z + u_r \partial_r u_z + u_z \partial_z u_z = -\frac{1}{\rho} \partial_z p, \tag{2.47}
\]

where \((u_r, u_\theta, u_z)\) denote the components of the velocity in a cylindrical referential. These equations are thus the Euler-equations in cylindrical coordinates where all azimuthal derivatives \(\partial_\theta\) are set to zero.

Since the flow, due to its axisymmetry, can be represented by its velocity-components in the \(r,z\) plane, and since incompressibility allows to eliminate one of the variables, the system can be completely described using two variables, the poloidal stream-function \(\psi\) (or toroidal vorticity \(\xi\)) representing the flow in the \(r-z\) plane (like the stream-function in 2D), and the azimuthal velocity, \(u_\theta\) (or angular momentum \(\sigma = ru_\theta\)).

In order to simplify as much as possible the equations, the definition of the vorticity, Laplacian and streamfunction are not completely standard. In [20] the following definitions were proposed, which will be used in this chapter. A streamfunction was defined related to the velocity-components in the poloidal plane such that,

\[
u_r = -\frac{1}{r} \partial_z \psi, \tag{2.48}\]

\[
u_z = \frac{1}{r} \partial_r \psi, \tag{2.49}\]

the angular momentum \(\sigma = ru_\theta\) is considered instead of the azimuthal velocity, and the vorticity \(\xi\) associated with the azimuthal component of the vorticity is defined by

\[
\xi = \frac{1}{r} (\partial_z u_r - \partial_r u_z) = \omega_\theta / r. \tag{2.50}\]

This vorticity can be computed from the streamfunction using the generalized Laplacian, \(\Delta_4\),

\[
\Delta_4 \psi \equiv \frac{1}{2y} \partial_z^2 \psi + \partial_r^2 \psi = -\xi, \tag{2.51}\]
where the variable $y$ is defined as $y = \frac{r^2}{2}$. The advantage of these definitions is that the equations (2.44) reduce to

\begin{align*}
\partial_t \sigma + \{ \psi, \sigma \} &= 0, \\
\partial_t \xi + \{ \psi, \xi \} &= \partial_z \left( \frac{\sigma^2}{4y^2} \right). 
\end{align*}

and where the Poisson brackets are defined classically (but with respect to the nonstandard variables $y$ and $z$), as

\[ \{ f, g \} = \partial_y f \partial_z g - \partial_z f \partial_y g \]

This system must be completed by Eq. (2.51). We have hereby defined a two-dimensional system, described by $\sigma, \xi$, as a function of the variables $y, z$.

Equations (2.52,2.53) provide several advantages. The first one is their compactness. The further-advantage is that it shows that the equation for $\sigma$ is a pure advection equation (like the vorticity in 2D). The most important property of this system, in the light of a development of a statistical mechanics theory, is that the formulation in terms of Poisson-brackets, and several algebraic manipulations [20] show that steady states of the axisymmetric Euler-equations correspond to the relations,

\begin{align*}
\{ \psi, \sigma \} &= 0, \\
\{ \psi, \xi \} + \{ \sigma, \sigma \} &= 0
\end{align*}

and the equations finally lead to

\begin{align*}
\sigma &= F(\psi), \\
\xi &= \frac{F(\psi)}{2\psi} F'(\psi) + G(\psi),
\end{align*}

where $F$ and $G$ are arbitrary functions. The details of the manipulations to derive expressions (2.57,2.58) can be found in [20].

### 2.2.2 Statistical mechanics of the axisymmetric system

As shown in [20], the axisymmetric Euler equations conserve, as for the 2D case, an infinite number of integral quantities. These quantities are the energy

\[ E = \frac{1}{2} \langle \xi \psi \rangle + \frac{1}{2} \left( \frac{\sigma^2}{r^2} \right), \]
all moments of the angular momentum,

\[ I_m = \langle \sigma^m \rangle, \tag{2.60} \]

and generalized helicities, defined by

\[ H_n = \langle \xi \sigma^n \rangle, \tag{2.61} \]

where \(\langle \ldots \rangle = \int r \, dr \, dz = \int dy \, dz\) is the volume integral over the cylindrical domain considered, normalized by \(2\pi\). As in 2D for the vorticity, the angular momentum is mixed, and it is therefore expected that a fine-scale structure is created for this quantity. Again we introduce a coarse-graining, by defining the probability density \(\rho(r, \eta, t)\) to have a value of \(\sigma \in [\eta, \eta + d\eta]\) at a position \(r\) and a time \(t\) in the unresolved scales. The coarse grained angular momentum is then obtained by

\[ \bar{\sigma}(r, t) = \int \rho(r, \eta, t) \eta \, d\eta. \tag{2.62} \]

In two dimensions, the coarse-graining consists in the introduction of a single density function, for the vorticity. In the present case, in addition to the function \(\rho\) describing the unresolved angular momentum, equations (2.52,2.53) also need the introduction of a coarse graining of the vorticity \(\xi\). However, it was proposed in [24, 25] to consider first the simpler case where the fluctuations of the angular momentum are sufficiently large compared to the vorticity fluctuations, to consider the latter frozen. This yields an easier problem to solve analytically, and some experimental support for this assumption was obtained in [22], as we will discuss in section 2.2.4. For this case, where the unresolved fluctuations in the vorticity are neglected, \(\bar{\xi} = \xi\), we can straightforwardly apply the ideas from the 2D case.

The first step is again the definition of a mixing entropy,

\[ S = -\int \rho \ln(\rho) d\eta dy dz, \tag{2.63} \]

and this entropy will be maximized, at the coarse-grained scale, under the constraint of conserved invariants, and a proper normalization of \(\rho\)

\[ \max \left\{ S|\bar{E}, \bar{H}_n, \bar{I}_m, \int \rho \, d\sigma = 1 \right\}. \tag{2.64} \]

This yields a variational equation of the form (2.33), involving an infinite number of constraints. As before, taking into account only the lower order constraints might give already a
first estimate of the functional relations (2.57,2.58). Doing this (retaining only $E$, $H_1$ and $I_1$), and using the assumption of frozen vorticity fluctuations $\bar{\xi} = \xi$, this corresponds to the variational equation

\[
\delta \int d\eta dydz(-\rho \ln \rho) - \frac{\beta}{2} \delta \int d\eta dydz\left(\xi \psi + \frac{\eta^2}{r^2}\right)\rho \\
- \mu \delta \int d\eta dydz\xi \eta \rho - \alpha \delta \int d\eta dydz \xi \rho - \int dydz\xi(r) \delta \int d\eta \rho = 0,
\]

(2.65)

where $\beta$, $\mu$, $\alpha$ and $\xi(r)$ are Lagrange multipliers. Variations in $\rho$ give

\[-1 - \ln(\rho) - \frac{\beta}{2} \frac{\eta^2}{r^2} - \mu \bar{\xi} \eta - \alpha \eta - \xi = 0\]

(2.66)
yielding the Gibbs state:

\[
\rho = \frac{1}{Z} \exp\left(-\frac{\beta}{2} \frac{\eta^2}{r^2} - (\mu \bar{\xi} + \alpha) \eta\right),
\]

(2.67)

where the factor $Z = \int \exp\left(-\frac{\beta}{2} \frac{\eta^2}{r^2} - (\mu \bar{\xi} + \alpha) \eta\right)$ normalizes the integral of $\rho$ to unity. Variations in $\bar{\xi}$ give

\[-\beta \psi - \mu \bar{\sigma} = 0.\]

(2.68)

From equation (2.68), the local average of $\sigma$ is:

\[
\bar{\sigma} = -\frac{r^2}{\beta} (\mu \bar{\xi} + \alpha).
\]

(2.69)

Looking more closely at these last two expressions, it is observed that this corresponds to a Beltrami flow ($\omega = \lambda u$, with $\lambda = -\beta/\mu$), if variations in angular momentum are neglected ($\alpha = 0$). Computing the variance of the fluctuations in angular momentum, this directly yields,

\[
\overline{\sigma^2} - \sigma^2 = \frac{2y}{\beta}
\]

(2.70)

so that

\[
\overline{u_\theta^2} - u_\theta^2 = \frac{1}{\beta}
\]

(2.71)

which relates the inverse temperature $\beta$ (in its statistical meaning) to the azimuthal velocity fluctuations, predicted to be uniform.
2.2.3 There are no stable maximum entropy states in axisymmetric turbulence, however ...

Up to this point the theoretical results presented in the last section seemed promising. However, further investigation [25] showed that the critical points corresponding to the most probable states discussed in the last section do neither correspond to maximum entropy states, nor to minimum states. Indeed, all critical points were shown to be saddle-points and thereby linearly unstable with respect to perturbations. This means that at long-times all the states corresponding to the critical points found above will be unstable in the presence of perturbations.

However, we do not consider infinitely long times in practice, and forcing and dissipation will introduce energy and other invariants in the system at a certain rate. It can therefore be that the saddle-points that were determined are stable for a certain time for a certain size of perturbations. This was investigated further in [25] using relaxation-equations based on maximum entropy production introduced for the case of 2D turbulence. It was shown that certain states were far more stable than others, and those metastable states qualitatively and, to a certain extent also quantitatively, resembled observations in the Von-Kármán experiment, which we will discuss now.

2.2.4 Observations in the Von Kármán experiment

The predictions of the theory were compared to results from the Von Kármán experiment (also called 'French washing-machine'), illustrated in figure 2.3. The flow in this device is generated by two counter-rotating impellers in a cylindrical vessel. The impellers can rotate with different speeds. [21] showed that in the high Reynolds number regime the time-averaged flow exhibits large-scale structures. As presented in Figure. 2.4, for different relative rotation speeds the flow can bifurcate from a monopolar to a dipolar structure. In addition to this, hysteresis in observed.

Based on the experimental configuration of Ravelet et al. [21], Monchaux et al. [22] set up an experimental case with a Reynolds number varying in the range \( Re = 10^2 - 3 \times 10^5 \), in order to check experimentally the theoretical predictions of Leprovost et al. [20]. They found that the two families of functions of stationary states in equation (2.57,2.58) can indeed characterize the mean flow, as shown in Figure. 2.5, even with the presence of finite viscosity and forcing. These functions are noted to be well-defined only in the portion of the flow far enough from the boundaries (wall and impellers), that is where the effects of forcing and dissipation are the lowest. Hence, experimental results show that the mean turbulent flow can be described by the functions \( F \) and \( G \). At high enough Reynolds number, these
Fig. 2.3 (Figure from [21]) The sketch of the 'French washing-machine', experimental setup of the von Kármán flow, generated by two counter-rotating impellers in a cylindrical vessel.

Fig. 2.4 (Figure from [21]) Dimensionless torque difference $\Delta K_p$ versus the rotation asymmetry $\theta$ of the impellers for $Re$ in the range $2 \times 10^5$ to $8 \times 10^5$, and the associated mean flow structures (torque measured on the impellers).
functions became linear, which means that the mean flow was in a Beltrami state, where the vorticity is locally aligned with velocity. The experimental mean flow was shown to exhibit bifurcation and hysteresis [21], two phenomena that can be also reproduced by the solutions of the statistical theory [25].

These observations are at the least surprising, a posteriori, since it was shown that no stable critical points exist in the theory, and since the experimental flow is not axisymmetric instantaneously, but only on the average. If the agreement with experiments turns out not to be fortuitous, it means that the axisymmetric saddle-point states are meta-stable and might determine the time-averaged mean flow, also in the presence of strong three-dimensional fluctuations.

### 2.3 The cascade of energy and other invariants

The power of statistical mechanics is its capability to predict equilibrium states without solving the governing equations which describe the dynamics of the system. In the case of turbulence this means that the Navier-Stokes equations do not need to be integrated in time to predict to what state an initial condition will evolve. The dynamics can however be very rich and particularly in a system as turbulence the complex spatiotemporal interactions among scales give rise to interesting phenomena. The universal description of these dynamics is not straightforward, but a fairly universal feature is the cascade of conserved quantities (invariants) through scales. We have seen that axisymmetric turbulence possesses an infinite number of invariants. Among them the invariants like $E$, $H_1$ and $I_1$ seem to be most important in a first-order description of the flow dynamics. What kind of cascade behavior the existence of these invariants implies is not clear from the outset, and this will be considered in chapter
5. This section gives a short overview of some notions related to the cascade for flows in two and three dimensions.

### 2.3.1 Energy and helicity cascades in three-dimensional turbulence

In natural turbulent flows, such as rivers and turbulent wakes, we may observe turbulent motions of different structure over a wide range of spatial scales. These motions were described as *eddies* of different sizes by Richardson[57] in 1922. He proposed that turbulence is an ensemble of eddies of different lengths scale \( l \) having a characteristic velocity \( u(l) \) and timescale \( \tau(l) \equiv l/u(l) \). Eddies in turbulence are then supposed to transfer energy from large scales to smaller and smaller scales until the dissipation will come start to act. Later on, Kolmogorov [58] formulated a set of hypotheses, which led to the dimensional prediction of the kinetic-energy spectrum,

\[
E(k) = C_E \varepsilon^{2/3} k^{-5/3},
\]

where \( C_E \) is supposed to be a universal constant named after Kolmogorov, \( \varepsilon \) the constant energy flux and \( k \) the wavenumber. Under the assumptions of Kolmogorov, this *inertial range* scaling should be observed at sufficiently high Reynolds numbers for a range of scales intermediate between the largest and the smallest lengthscales. This energy spectrum is remarkably well verified in a wide range of high Reynolds number flows.

In addition to energy, helicity \( H_1 \) is another inviscid invariant of three-dimensional homogeneous turbulence. It plays an important role in the spontaneous generation of planetary magnetic fields [59]. Even in the absence of magnetic fields, helicity, as one of the two unique invariants of the three-dimensional Navier-Stokes equations, has obviously been the center of a large number of investigations. The cascade of helicity was first discussed by Brissaud *et al.* [60]. They concluded that there were two possibilities for the nature of the helicity cascade. Either the forward cascades of energy and helicity are coexisting, or there will be a forward cascade of helicity accompanied by an inverse cascade of energy. The latter possibility, where a "pure" helicity cascade is observed in a range of scales where the energy is not transferred, would correspond to a helicity spectrum of the form

\[
H_1(k) \sim \delta^{2/3} k^{-4/3},
\]

and such a process, inhibiting the forward cascade of energy would imply in the same range of scales an energy spectrum of the form,

\[
E(k) \sim \delta^{2/3} k^{-7/3}. \tag{2.74}
\]
However, such a scaling is not observed, most probably due to the fact that helicity is not a definite positive quantity like the energy \cite{43}. This was first illustrated by numerical calculations in reference \cite{61}. These simulation supported the first possibility, that the energy and helicity forward cascade simultaneously with a similar scaling behavior

\[ H_1(k) \sim \delta \varepsilon^{-1/3} k^{-5/3}, \tag{2.75} \]

where \( \delta \) is the rate of transfer of helicity, with an energy spectrum which remains of the form (2.72). It seems that global helicity does not drastically change the nature of the energy cascade. However, helicity can also be observed locally and an arbitrary velocity field can be decomposed as a function of helical modes \cite{42}. This decomposition was for instance used to study the influence of helicity using shell-models \cite{62–64}.

The fact that the helicity is a composition of both positive and negative values allows a further investigation of the structure of turbulent transfer. In order to see the quantities with specific sign of helicity, the spectra of energy and helicity can be decomposed into the contributions associated to the positive and negative helical modes, leading to \cite{43, 64},

\[ E_{\pm}(k) \sim \frac{1}{2} C_E \varepsilon^{2/3} k^{-5/3} \left[ 1 \pm \frac{\gamma}{2k} \left( \frac{\delta}{\varepsilon} \right) \right], \tag{2.76} \]

which shows that the influence of helicity on the energy spectrum vanishes for large wavenumbers. For the helicity spectrum this yields,

\[ H_{1\pm}(k) \sim C_E \varepsilon^{2/3} k^{-2/3} \left[ 1 \pm \frac{\gamma}{2k} \left( \frac{\delta}{\varepsilon} \right) \right], \tag{2.77} \]

with \( \gamma = C_{H_1}/C_E \). The further use of this decomposition recently led to the first observation of an inverse energy cascade in three-dimensional homogeneous and isotropic turbulence \cite{44}. By considering only one type of helical mode in the numerical simulation, it was observed that the local cascade behavior of energy corresponded to an inverse energy cascade with \( E(k) \sim k^{-5/3} \) and a direct helicity cascade \( E(k) \sim k^{-7/3} \), as proposed as a possibility for the cascade of helicity in the early work of Brissaud et al. \cite{60}.

### 2.3.2 Energy spectra in two-dimensional turbulence

Kraichnan \cite{65, 66} and Batchelor \cite{55} predicted the shape of the energy spectra for two-dimensional turbulence. Indeed, in his work, Kraichnan predicted the scenario of a double cascade \cite{65} for the case of an isotropic two-dimensional turbulent flow sustained by an external forcing acting on a specific wavenumber \( k_f \). In this case the kinetic energy will
cascade towards small wavenumbers ($k < k_f$) and the enstrophy-transfer will dominate the large wavenumbers ($k > k_f$). Dimensional analysis, using the local-energy-transfer assumption, leads then to the inverse cascade energy spectrum of the form

$$E(k) \sim \varepsilon^{2/3}k^{-5/3}, \quad (2.78)$$

and the direct cascade of enstrophy is associated with the energy spectrum

$$E(k) \sim \eta^{2/3}k^{-3}, \quad (2.79)$$

with possible logarithmic corrections [66]

$$E(k) \sim \eta^{2/3}k^{-3}\left[\ln(k/k_{\text{min}})\right]^{-1/3}. \quad (2.80)$$

Here $\eta$ is the flux rate of enstrophy. These dual cascade predictions have been intensively studied by both experiments and numerical efforts and high-resolution simulations [67] corroborate the predictions of a co-existence of the two inertial ranges. Earlier observations of the inverse cascade in 2D turbulence in numerical simulations are reported in [68–71]. Later, larger simulations of the inverse cascade have provided convincing evidence of Kolmogorov scaling. Further recent DNS of the inverse cascade [40, 41] are also consistent with these scaling results with added insight into the mechanism of the inverse cascade. These results showed that the merging of vorticity contributes little to the inverse energy flux. Anomalous scaling at small wavenumbers, different from classical predictions, is also observed. For instance, anomalous $k^{-3}$ scaling at scales where the inverse cascade is expected can arise due to finite size effects [72] or due to the presence of hypofriction [73], both leading to energy condensation at large scales [65].

As for the direct cascade in two-dimensional turbulence, the physical mechanism is associated with vortex-gradient stretching, induced by the large-scale vortices near the injection scale, generating fine vortex filaments, in which the viscosity will dissipate the vorticity. Nevertheless, early DNS reported results very different from the $k^{-3}$ predictions, in which the slope of the spectra is much steeper. This deviation is observed both in decaying simulations [74–76] and in forced ones [77, 78]. Whereas in the forced simulations the limited resolution is probably responsible for these deviations, in the decaying case, other mechanisms can be proposed. It was suggested in these latter two references that the steeper cascade could be correlated with the presence of long-living, strong vortices,
2.3 The cascade of energy and other invariants

2.3.3 Influence of intermittency on the scaling of energy spectra

The scaling proportional to $k^{-5/3}$ for energy spectra in 3D and in the inverse cascade range of 2D turbulence is in general well verified. The influence of strongly non-Gaussian small scale activity seems small in general on the scaling exponent related to the energy cascade range. Basdevant et al. [77], proposed a simple phenomenological model basically derived from ideas of Frisch et al [79] in order to consider intermittency on the scaling theory of spectra in the forward enstrophy range and to explain the very steep spectra observed in both experiments and simulations.

It is stressed here that the term intermittency has several meanings in turbulence. The one here is the tendency of the probability distributions of certain quantities to develop long tails corresponding to very strong events, leading to high values of the Kurtosis. In the proposition of Basdevant et al., the influence of both space and time intermittency on the scaling of the forward enstrophy spectrum are discussed. For space intermittency, they claim that the energetic, active structures of wavenumber $k$ are confined in physical space in a limited domains $D(k)$, and they assume that these domains are permanently excited and embedded in one another (in the fashion of a Richardson cascade), continuously interacting with each other. To exclusively represent the energy spectrum of these excited domain $D(k)$, an active energy spectrum $E^*(k)$ is defined. Moreover, the local eddy turnover time $\tau(k)$ is argued to mainly depends on these active structures,

$$\tau(k) \sim Z^*(k)^{-1/2}, \quad (2.81)$$

where

$$Z^*(k) = \int_0^k k' Z^*(k') dk'. \quad (2.82)$$

is the active enstrophy integral. Meanwhile the enstrophy cascade rate $\eta$ depends on both the observed spectrum and the eddy-turnover time

$$\eta = k^3 E(k)/\tau(k). \quad (2.83)$$

In order to evaluate the influence of this space-intermittency on the spectra, they assume a self-similar structure, where the area of $D(k)$ is proportional to $k^\epsilon$, therefore

$$E^*(k) \sim k^\epsilon E(k). \quad (2.84)$$
By solving the equations above, one obtains the prediction for intermittent enstrophy inertial range as $\varepsilon > 0$,

\begin{align}
E^*(k) &\sim k^{-3+(2\varepsilon/3)}, \quad (2.85) \\
E(k) &\sim k^{-3-(\varepsilon/3)}, \quad (2.86)
\end{align}

and the non-intermittent solution with $\varepsilon = 0$ is,

\begin{align}
E^*(k) &= E(k) \sim k^{-3}(\ln k)^{-1/3}, \quad (2.87)
\end{align}

which leads to the Kraichnan’s spectra law with logarithmic correction. Nevertheless, they also noted that the exponent characterizing space intermittency $\varepsilon$ should be within $0 \leq \varepsilon \leq 2$. Since the fraction of space $k^\varepsilon$ occupied by the active domain should at least contain one structure, the area of this structure is at least of order $k^{-2}$. Then it follows that space intermittency by itself is not able to explain the steeper spectra obtained in their simulations. Subsequently they consider temporal intermittency, by assuming that $D(k)$ also has a probability of occurrence in time proportional to $k^{\varepsilon'}$, $\varepsilon' \geq 0$. And the same argument using the new active spectra $E^{**}(k)$ yields

\begin{align}
E^{**}(k) &= k^{\varepsilon + \varepsilon'} E(k), \quad (2.88)
\end{align}

which gives

\begin{align}
E^{**}(k) &\sim k^{-3+\left[2(\varepsilon+\varepsilon')/3\right]}, \quad (2.89) \\
E(k) &\sim k^{-3-\left[(\varepsilon+\varepsilon')/3\right]}, \quad (2.90)
\end{align}

for $\varepsilon > 0$ or $\varepsilon' > 0$. These last two equations were argued to be consistent with their numerical results.

Basdevant et al. showed thus how spatial and temporal intermittency may affect the spectra. Later on, in an investigation of McWilliams et al. [75], Basdevant’s work was further discussed and it was argued that the Kurtosis of vorticity can reflect a particular kind of spatial and temporal intermittency, related to the formation of isolated coherent structures. These ideas were supported by a subsequent investigation [80], where it was shown that by filtering the vortices from a vorticity field (removing vorticity larger than a threshold), one can recover the $k^{-3}$ prediction by measuring the energy spectra of the remaining background field.
2.3 The cascade of energy and other invariants

2.3.4 Wavenumber spectra in Von Kármán flow

All derivations in the previous sections assumed the velocity field to be isotropic. If this is not the case, what can be expected? Herbert et al. [38] studied the energy spectra in the von Kármán experiments that we mentioned previously, for the first time without using the Taylor hypothesis, and tested hypotheses on the locality of the transfer mechanism. Indeed, in the experiments spectra of the azimuthal energy were observed which were not incompatible with a dual cascade mechanism. This is not in agreement with the observations in three-dimensional isotropic simulations (as discussed before [43, 61]) where a single cascade direction for both energy and helicity was observed. This single direction is possible since helicity is not a positive-definite quantity like the enstrophy in two-dimensional turbulence. However, if the flow is dominantly Beltrami, the helicity becomes a quantity which is of a single sign. This might lead to the blocking of the energy cascade (the alternative scenario proposed in reference [60]) and thereby a dual cascade.

Even if we assume a double cascade, the energy spectra can vary according to the assumptions on the locality of the energy transfer. If, as in Kolmogorov’s picture, the transfer is due to local straining, the straining time-scale will be proportional to \( \tau_{loc} \sim \left( \int p^2 E(p) dp \right)^{-1/2} \). However, the presence of a mean-flow can introduce a dominant straining time related to the large-scale shear \( \tau_{non-loc} \sim S^{-1} \).

For the direct helicity cascade, the local estimate leads, combined with a conserved flux assumption, to

\[
E^{loc}(k) \sim \delta^{2/3} k^{-7/3},
\]

and if the nonlocal shearing is dominant in the transfer, to

\[
E^{non-loc}(k) \sim \delta S^{-1} k^{-2}.
\]

For the inverse energy cascade, the local and non-local results are,

\[
E^{loc}(k) \sim \varepsilon^{2/3} k^{-5/3},
\]

\[
E^{non-loc}(k) \sim \varepsilon S^{-1} k^{-1}.
\]

The experiments seem to be compatible with dual cascade behavior, with the inverse energy cascade compatible with \( E(k) \sim k^{-1} \) (non-local cascade) and direct helicity cascade compatible with \( E(k) \sim k^{-2} \) (non-local) at low Reynolds numbers, and with \( E(k) \sim k^{-2} \) (non-local) or \( k^{-7/3} \) (local) at higher Reynolds numbers.
2.3.5 In axisymmetric turbulence

In purely axisymmetric turbulence, no cascade behavior has ever been measured. However, in the three-dimensional turbulence of the von Kármán experiment mentioned above, a dual cascade similar to the two-dimensional case seemed to be compatible with the data. These interesting results lead naturally to the question about what kind of cascade behavior will be present in the strict axisymmetric turbulence simulations we will consider in the following. The ideas in this section give us already some ingredients to base possible explanations on.

2.4 Discussion

In this chapter, we reviewed the theory which will be relevant for the remainder of this manuscript. In particular, some of the questions that we will try to answer in the following are: are the predictions of statistical mechanics for axisymmetric turbulence in agreement with numerical results obtained from integrating the axisymmetric Navier-Stokes equations? Is a Beltrami flow observed? Are selective decay arguments relevant? Are the energy spectra in agreement with observations in the von Kármán experiment?

In order to do all this, a fully spectral method will be used to numerically investigate both freely decaying and forced axisymmetric turbulence, for a variety of initial conditions.
Chapter 3

Numerical methods

This chapter introduces the numerical methods which are used in this thesis. The objective is to simulate axisymmetric turbulence in a cylindrical domain, periodic along the axial direction, and to satisfy a no-penetration boundary condition at the wall. In Figure 3.1 we show a sketch of this domain. The radius of the cylinder is called \( a \) and the axial length is denoted by \( L_z \). Within this domain, we use a fully spectral method based on the Eigenfunction of the curl, inspired by the work of Montgomery and collaborators [34–36], which was implemented and used to simulate magnetohydrodynamic (MHD) turbulence and two-dimensional turbulence. Our code has been developed starting from a two-dimensional version, which was kindly provided to us by Dr. Shuojun Li. The details of the code that we modified in order to study the axisymmetric problem can be found in references [31–33], where the focus was on the decay and relaxation properties in 2D turbulence.

Fig. 3.1 The computational domain.
3.1 The spectral method

3.1.1 Eigenfunction of the Curl

The expression for the eigenfunctions of the curl in cylindrical coordinates \((r, \theta, z)\) was first derived by Chandrasekhar and Kendall [27]. In their first work on this subject, they studied force-free magnetic fields. Indeed, physical systems which are left to relax will often tend to self-organize to a system where the internal forces are as weak as possible. In the description of MHD, the force on the fluid, associated with the interaction of a magnetic field \(B\) with the induced current \(J\) in the fluid, is given by the Lorentz-force. This force which appears on the right hand side of the Navier-Stokes equations is given by

\[
F_L = J \times B
\]  

(3.1)

where the current is \(J = \mu_0^{-1} \nabla \times B\), with \(\mu_0\) the magnetic permeability of vacuum. In order to have a force-free, but non-zero magnetic field, we should therefore have,

\[
(\nabla \times B) \times B = 0,
\]

(3.2)

and a non-trivial solution is

\[
\nabla \times B = \lambda B,
\]

(3.3)

where \(\lambda\) is a constant. Taking the curl of this expression, we have,

\[
\nabla(\nabla \cdot B) - \nabla^2 B = \lambda^2 B
\]

(3.4)

and since a magnetic field is necessarily solenoidal, \(\nabla \cdot B = 0\), this leads to

\[
\nabla^2 B + \lambda^2 B = 0
\]

(3.5)

which is the vector-wave equation, and the solution of equation (3.3) is among the solutions of equation (3.5). By studying the solutions of equation (3.5), Chandrasekhar and Kendall showed that a general expression of the solutions of equation (3.3) can be written as

\[
B = \frac{1}{\lambda} \nabla \times \nabla \times (a \psi_H) + \nabla \times (a \psi_H),
\]

(3.6)

where \(a\) is a fixed unit vector and \(\psi_H\) is a scalar function which is the solution of the Helmholtz equation:

\[
(\nabla^2 + \lambda^2) \psi_H = 0.
\]

(3.7)
In cylindrical coordinates, a natural choice for $a$ is $e_z$.

The eigenfunction of the curl was proved useful in expanding not only $B$ but also the velocity field $v$ in plasmas [34–36, 81–83]. The expression of such eigenfunction for $v$ is equivalent to the expression of the magnetic field in equation (3.6), which will be presented in more detail in the following. In references [35, 36], these eigenfunctions of the curl were used as an orthogonal basis in which an incompressible velocity field $v(r, \theta, z, t)$ was expanded as:

$$v(r, \theta, z, t) = \sum_{nmq} \xi^v_{nmq}(t) A_{nmq}(r, \theta, z),$$

(3.8)

where $\xi^v_{nmq}$ are complex coefficients that depend on time, and $A_{nmq}(r, \theta, z)$ are the Chandrasekhar-Kendall (CK) eigenfunctions:

$$A_{nmq} = I^{-\frac{1}{2}}_{nmq} [\lambda_{nmq} \nabla \psi_{nmq} \times e_z + \nabla \times (\nabla \psi_{nmq} \times e_z)],$$

(3.9)

$$(\nabla^2 + \lambda_{nmq}^2) \psi_{nmq} = 0,$$

(3.10)

in which $n, m, q$ represent the modal indexes for physical space $r, \theta, z$, respectively. $I^{-\frac{1}{2}}_{nmq}$ is a normalisation coefficient ensuring that

$$\frac{1}{V} \int_V r \, dr \, d\theta \, dz \, A_{nmq}(r, \theta, z) A^*_{nmq}(r, \theta, z) = 1,$$

(3.11)

In which $V$ is the volume of domain. As above, $\lambda_{nmq}$ are constants and $\psi_{nmq}$ are scalar function solutions of the Helmholtz equation (3.10) whose exact expressions depend on the boundary conditions and symmetries. Using expression (3.3), the vorticity field can be easily expressed as:

$$\omega(r, \theta, z, t) = \sum_{nmq} \lambda_{nmq} \xi^v_{nmq}(t) A_{nmq}(r, z).$$

(3.12)

A series of papers, by Yoshida et al. [28–30], established a rigid mathematical background for this kind of expansion.

In the following section we will derive the expressions of $\lambda_{nmq}$, $\psi_{nmq}$ and $A_{nmq}$ for axisymmetric flows in a cylindrical domain with periodic boundary conditions in the axial direction and non-penetration boundary conditions in the radial direction.
3.1.2 Expansion of axisymmetric velocity fields in terms of Chandrasekhar-Kendall modes

It will be thereafter assumed that $\frac{\partial}{\partial \theta} = 0$. With this property, we start with a new notation of index of the scalar equation as $\psi_{nq}$, which indicates that it is independent of $\theta$. By the method of separation of variables, the trial solutions of equation (3.10) can be written as:

$$\psi_{nq}(r, z) = R(r)Z(z), \quad (3.13)$$

where $R$ and $Z$ are scalar functions of the radial and axial coordinates, respectively. Inserting this expression in equation (3.10) leads to:

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \lambda_{nq}^2 = -\frac{Z''}{Z}, \quad (3.14)$$

in which the left hand side (respectively right hand side) depends on $r$ (respectively $z$) only. As a consequence, they must be equal to a constant, that we will denote $\alpha_{nq}$. The functions $R$ and $Z$ are therefore solutions of

$$R'' + \frac{1}{r} R' + \lambda_{nq}^2 R = \alpha_{nq} R, \quad (3.15)$$

$$Z'' + \alpha_{nq} Z = 0. \quad (3.16)$$

If $Z(z)$ is considered periodic, then $\alpha_{nq}$ must be positive. We will denote it as $\alpha_{nq} = k_n^2$. The general solution of equation (3.16) then reads

$$Z(z) = A e^{ik_n z} + B e^{-ik_n z}, \quad (3.17)$$

where $A$ and $B$ are constants, and $k_n$ is the Fourier wavenumber. The differential equation satisfied by $R$ is now

$$R'' + \frac{1}{r} R' + (\lambda_{nq}^2 - k_n^2)R = 0, \quad (3.18)$$

which is well known as the Bessel’s differential equation, the general solution of which is

$$R(r) = C J_0(\gamma_{nq} r), \quad (3.19)$$

where $J_0$ is the Bessel function of the first kind with the order 0, and $\gamma_{nq}$ is a constant satisfying the relation

$$\gamma_{nq}^2 = \lambda_{nq}^2 - k_n^2. \quad (3.20)$$
In consequence, in cylindrical axisymmetric field, the explicit form of \( \psi_{nq} \) is

\[
\psi_{nq}(r,z) = J_0(\gamma_{nq} r)e^{-ik_nz}
\] (3.21)

where \( k_n = 2\pi n/L_z \) \((n = 0, \pm 1, \pm 2, \ldots)\). \( \gamma_{nq} \) are assumed to be always positive \((q = 1, 2, 3, \ldots)\), and both positive and negative values of \( \lambda_{nq} \) must be allowed, as will be shown later. Boundary conditions will be used to determine the \( \gamma_{nq} \) parameters, as will be explained in section 3.1.3. The values of \( \lambda_{nq} \) will be deduced from those of \( k_n \) and \( \gamma_{nq} \) through equation (3.20).

With simple calculation, equation (3.9) can be rewritten as

\[
A_{nq} = I_{nq}^{-\frac{1}{2}} \left[ \begin{array}{c} 0 \\
\lambda_{nq} \frac{\partial \psi_{nq}}{\partial r} \\
0 \end{array} \right] + \left[ \begin{array}{c} 0 \\
\frac{1}{r} \frac{1}{r} \frac{\partial \psi_{nq}}{\partial r} \\
\frac{1}{r} \frac{\partial \psi_{nq}}{\partial r} \\
\end{array} \right] \right],
\] (3.22)

in which we have underlined the fact that the term \( \lambda_{nm} \nabla \psi_{nm} \times e_z \) represents the azimuthal (toroidal) velocity and the other term \( \nabla \times (\nabla \psi_{nm} \times e_z) \) represents the radial and vertical (poloidal) velocity components. Substituting equation (3.21) in (3.22), and using the properties of the Bessel functions [84], finally we have

\[
A_{nq} = I_{nq}^{-\frac{1}{2}} \left[ \begin{array}{c} k_n \gamma_{nq} I_1(\gamma_{nq} r)e^{-ik_nz} \\
\lambda_{nq} \gamma_{nq} I_1(\gamma_{nq} r)e^{-ik_nz} \\
\frac{\gamma_{nq}^2}{I_2(\gamma_{nq} r)} e^{-ik_nz} \end{array} \right].
\] (3.23)

This is the detailed expression for the CK eigenfunctions in our method. We rewrite the expansions for our system,

\[
v(r,z,t) = \sum_{nq} \xi_{nq}^v(t) A_{nq}(r,z)
\] (3.24)

\[
\omega(r,z,t) = \sum_{nq} \lambda_{nq} \xi_{nq}^\omega(t) A_{nq}(r,z).
\] (3.25)

Some simple characteristics of the flow can be recovered from equation (3.23). In particular, the divergence of vector \( A_{nq} \)

\[
\nabla \cdot A_{nq} = \frac{1}{r} \frac{\partial (rA_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = 0,
\] (3.26)
which trivially agrees with the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. By taking the curl of the vector
\[
\nabla \times \mathbf{A}_{nq} = \lambda_{nq} \mathbf{A}_{nq},
\]
(3.27)
which corresponds to the equation (3.12), one recovers that each mode is Beltrami with a proportionality coefficient between $\omega$ and $v$ equal to $\lambda_{nq}$.

Finally, by writing the complex coefficients \( \xi_{nq}(t) = a_{nq}(t) + ib_{nq}(t) \), where \( a_{nq}(t) \) and \( b_{nq}(t) \) are real functions of time, we are able to derive the precise formula of the velocity field:
\[
\begin{align*}
    u_r &= \sum_{n,q} \xi_{nq} I_q^{-1} i k_n \gamma_{nq} J_1(\gamma_{nq} r) e^{-i k_n z} \\
    &= \sum_{n>0,q} I_q^{-1} k_n \gamma_{nq} J_1(\gamma_{nq} r)[2a_{nq}\sin(k_n z) - 2b_{nq}\cos(k_n z)] \\
    u_\theta &= \sum_{n,q} \xi_{nq} I_q^{-1} \lambda_{nq} \gamma_{nq} J_1(\gamma_{nq} r)e^{-i k_n z} \\
    &= \sum_{n>0,q} I_q^{-1} \lambda_{nq} \gamma_{nq} J_1(\gamma_{nq} r)[2a_{nq}\cos(k_n z) + 2b_{nq}\sin(k_n z)] + \sum_q I_{0q}^{-1} \lambda_{0q} \gamma_{0q} J_1(\gamma_{0q} r)a_{0q} \\
    u_z &= \sum_{n,q} \xi_{nq} I_q^{-1} \gamma_{nq}^2 J_0(\gamma_{nq} r)e^{-i k_n z} \\
    &= \sum_{n>0,q} I_q^{-1} \gamma_{nq}^2 J_0(\gamma_{nq} r)[2a_{nq}\cos(k_n z) + 2b_{nq}\sin(k_n z)] + \sum_q I_{0q}^{-1} \gamma_{0q}^2 J_0(\gamma_{0q} r)a_{0q}.
\end{align*}
\]
(3.28)

So far we have shown the basic mathematical details of the expansions of the velocity and vorticity fields in terms of the CK eigenfunctions. In the following section we will focus on how to determine the values of the $\gamma_{nq}$ and $\lambda_{nq}$ parameters.

### 3.1.3 Parameter determination

**Determination of $\gamma_{nq}$**

The values of $\gamma_{nq}$ can be determined by applying the proper boundary conditions.

When $k_n \neq 0$, the no-penetration condition on the wall, $\mathbf{v}(r = a) \cdot \mathbf{e}_r = 0$, yields
\[
J_1(\gamma_{nq} a) = 0,
\]
(3.29)
which determines an infinite sequence of positive $\gamma_{nq}$: they are the zeros of the Bessel function $J_1(ax)$. 

Due to orthogonality of the axial and radial direction, this condition is independent of \( n \), for \( n > 0 \). For \( n = 0 \), using expression (3.29) implies a zero value for the component \( u_\theta \) at the wall for a flow invariant along \( z \). This is also a reasonable condition. We therefore use (3.29) for all values of \( n \), including 0. \( \gamma_{nq} \) is therefore independent of \( n \) and we drop this index: \( \gamma_{nq} \rightarrow \gamma_q \).

This numerical method does not allow to impose no-slip on the wall. This can be seen from equation (3.23). Since \( J_1(x) \) and \( J_0(x) \) do not have any coinciding zeros, our method does not allow to impose zero values for all three velocity components on the wall. We have chosen \( u_r \) to be zero at the wall, which implies \( u_\theta = 0 \), but leaves \( u_z \) free at the boundary. This must be seen as one inevitable disadvantage of the present method. Equation (3.29) can be used to determine all \( \gamma_q \) parameters.

In addition to \( u_r(a) = u_\theta(a) = 0 \), this choice implies a condition on the axial velocity condition. It can be shown that \( J_1(\gamma_q a) = 0 \) implies that the mean flux through the cylinder is zero.

\[
\int \mathbf{e}_z \cdot \mathbf{A}_{nq} \, d^3x = 2\pi L_z L_{nq}^{-\frac{1}{2}} \int_0^a \gamma_q^2 J_0(\gamma_r) r \, dr = 2\pi L_z L_{nq}^{-\frac{1}{2}} \left[ \gamma_q r J_1(\gamma_r) \right]_0^a = 2\pi L_z L_{nq}^{-\frac{1}{2}} \gamma_q a J_1(\gamma_q a).
\]
Therefore, the fluxless condition is also satisfied if one imposes:

\[
J_1(\gamma_q a) = 0.
\] (3.31)

**Determination of \( \lambda_{nq} \)**

Once the \( k_n \) and \( \gamma_{nq} \) have been determined for \( n = 0, \pm 1, \pm 2, \ldots \) and \( q = 1, 2, 3, \ldots \), one still needs to determine the \( \lambda_{nq} \) parameters. In fact it can be easily noticed that the relation:

\[
\lambda_{nq}^2 = \gamma_q^2 + k_n^2,
\] (3.32)
trivially deduced from equation (3.20), allows to determine only the absolute value of \( \lambda_{nq} \), not its sign. As indicated by former analytical works [28–30], for a positive eigenvalue \( \lambda_{nq} \), there is also a negative eigenvalue. For each value of \((n,q)\), the two signs must be accounted for in order to establish a complete basis of eigenfunctions. Here we show a simple complementary test to confirm that.

We consider a simple axisymmetric and incompressible flow in a cylindrical domain, which satisfies periodic boundary conditions in the axial direction and no-penetration along
the wall:

\[ u_r(r, z) = -\frac{1}{r} \partial_z \psi(r, z) \]

\[ u_\theta(r, z) = \sin^3 \left( \frac{2\pi z}{L_z} \right) (r - a)^2 r \tag{3.33} \]

\[ u_z(r, z) = \frac{1}{r} \partial_r \psi(r, z), \]

where

\[ \psi(r, z) = \cos^2 \left( \frac{2\pi z}{L_z} \right) (r - a)^3 r. \tag{3.34} \]

The idea is to first transfer this velocity field from physical space to spectral space, and then transfer back to the physical space to see if the resulting field is similar to the initial one or not. Three types of trial basis are tested: the eigenfunctions with all signs of \( \lambda_{nq} \), the eigenfunctions with positive \( \lambda_{nq} \) only, and the eigenfunctions with negative \( \lambda_{nq} \) only.

In Figure. 3.2(a), we have the flow field of the three components plotted in a computational domain \((r - z)\), with \( R = 2\pi \) and \( L_z = 2\pi \). By transferring this field into spectral space and then transfer it back to physical space with both eigenfunctions of \( \pm \lambda_{nq} \), we obtain Figure. 3.2(b) and this flow field is noted as Field 1. We see that Figure. 3.2(a) and Figure. 3.2(b) are qualitatively identical. Then when the original field is transferred back to physical space taking only into account the eigenfunctions of either \( + \lambda_{nq} \) or \( - \lambda_{nq} \), we note the flow fields as Field 2 and Field 3 respectively. These two fields are illustrated in Figure. 3.2(c) and Figure. 3.2(d). It is obvious that these two Figures are different from the original field in Figure. 3.2(a). Moreover, we have also calculated the total energy of these 4 fields in Table. 3.1. It shows that the total energies of the original field and Field 1 are the same, which means no lose of information during this transfer. Contrarily, Field 2 and Field 3 are just half of the original total energy, which implies clearly a loss of information during theirs transfers. In all, by all the tests described above, we conclude that the basis must contain both signs of \( \lambda_{nq} \) to be complete.

<table>
<thead>
<tr>
<th>Original Field</th>
<th>Field 1</th>
<th>Field 2</th>
<th>Field 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2256</td>
<td>2256</td>
<td>1128</td>
<td>1128</td>
</tr>
</tbody>
</table>

Table 3.1 Total energy of different test fields. We did note the small difference in order of \( 10^{-5} \) in energy, it is due to the projection procedure, which is not part of the numerical method.
3.1 The spectral method

(a) The original field proposed in equation (3.33).
(b) Field 1: reconstruction of the original field with all the eigenfunctions of $\pm \lambda_{nq}$.
(c) Field 2: Reconstruction of the original field with only the eigenfunctions of $+\lambda_{nq}$.
(d) Field 3: Reconstruction of the original field with only the eigenfunctions of $-\lambda_{nq}$.

Fig. 3.2
Discussion of the boundary conditions

Now that the parameters determined, we want to point out that the boundary conditions here are in a mixed situation. As mentioned above, the no-penetration and fluxless conditions demand that $J_1(\gamma_{nq}a) = 0$. By checking the velocity expansion in equations (3.28), we find that the component $u_r$ and $u_\theta$ naturally vanish at the boundary. In the contrary, the axial component $u_z$ is not necessarily zero at the wall, because the function $J_0(\gamma_{nq}a)$ is non-zero. Therefore, the choice of our parameter will lead to the no-slip condition in the azimuthal direction but a free slip in the axial direction. This short-coming was also mentioned in the article of Shan et al. [36]. This explains our choice of fluxless vertical conditions in the determination of $\gamma_0 q$, so that the slip boundary of $u_z$ won’t cause any significant vertical mean flow development. It is necessary for us to clarify that unfortunately total no-slip conditions are not compatible using this expansion in our work, but it constitutes an interesting future subject to improve this method.

3.1.4 Discretization of the Navier-Stokes equation

We finally reformulate in this subsection the Navier-Stokes equations in the basis of CK eigenfunctions. For this, we substitute the expansions (3.8) and (3.12) into the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} = (\mathbf{v} \times \omega) - \nabla p + \nu \nabla^2 \mathbf{v},$$

and then take the scalar product with the individual $A_{nq}$. This operation converts the Navier-Stokes equation into a set of ordinary differential equations for the expansion coefficients:

$$\frac{\partial}{\partial t} \xi_{nq}^v = \frac{1}{V} \int_V A_{nq}^* \cdot \left[ (\mathbf{v} \times \omega) - \nabla p + \nu \nabla^2 \mathbf{v} \right] d^3 x. \quad (3.36)$$

Interestingly, the pressure term drops out since:

$$\int_V A_{nq}^* \cdot (-\nabla p) d^3 x = \int_V -\nabla (p A_{nq}^*) d^3 x + \int_V p \nabla \cdot A_{nq}^* d^3 x = 0 \quad (3.37)$$

($p$ is periodic along the z-axis and the $A_{nq}$ modes are divergence-free: the incompressibility condition is therefore accounted for implicitly since the $A_{nq}$ modes are divergence-free), and equation (3.36) can be rewritten as:

$$\frac{\partial}{\partial t} \xi_{nq}^v = \sum_{n' q' n'' q''} \lambda_{n' q' n'' q''} \xi_{n' q'}^v \xi_{n'' q''}^v \cdot \frac{1}{V} \int_V rdrdzd\theta A_{nq}^* \cdot (A_{n' q'} \times A_{n'' q''}) - \nu \lambda_{nq}^2 \xi^v_{nq} \quad (3.38)$$
After some algebra, the term \( \frac{1}{V} \int_V \mathbf{A}_{nq}^* \cdot (\mathbf{A}_{n'q'} \times \mathbf{A}_{n''q''}) d^3x \) can be written as:

\[
\frac{1}{V} \int_V \mathbf{A}_{nq}^* \cdot (\mathbf{A}_{n'q'} \times \mathbf{A}_{n''q''}) = \frac{1}{V} \int_V i \cdot I_{nq}^{1/2} \cdot I_{n'q'}^{1/2} \cdot e^{-i(k_{n''}+k_{n'}-k_n)z} \cdot \{ -k_n \gamma_q J_1(\gamma_q r) \cdot [\lambda_{n'q'} \gamma_{q'} J_1(\gamma_{q'} r) \cdot \gamma_{q''}^2 J_0(\gamma_{q''} r) - \gamma_q^2 J_0(\gamma_q r) \cdot [\lambda_{n''q''} \gamma_{q''} J_1(\gamma_{q''} r)] + \lambda_{n'q'} \gamma_{q'} J_1(\gamma_{q'} r) \cdot [\lambda_{n''q''} \gamma_{q''} J_1(\gamma_{q''} r) - \lambda_{n'q'} \gamma_{q'} J_1(\gamma_{q'} r) \cdot \gamma_{q''}^2 J_0(\gamma_{q''} r)] ] \} rdr \theta dz. \tag{3.39}
\]

In this approach the Navier-Stokes equation is therefore solved by integrating equation (3.38), in which the nonlinear term is evaluated through equation (3.39). The viscous term is left on the right side of the equation and is integrated explicitly with the \( \xi_{nq}^\nu \) of the previous time step.

### 3.1.5 Discretization of the inviscid invariants

As noted by former theoretical contributions [20, 24], axisymmetric inviscid flows admit an infinite number of conserved quantities. In addition to the total energy, there are also the Casimirs

\[
I_G = \int G(\sigma) dydz \tag{3.40}
\]

and the generalized helicities

\[
H_F = \int \xi F(\sigma) dydz \tag{3.41}
\]

where \( G \) and \( F \) are any regular functions, \( y = \frac{r}{\tau} \), \( \sigma = ru_\theta \), \( \xi = \frac{\omega_0}{\tau} \). We introduce the notation \( I_n \) and \( H_m \) for the case where \( F \) or \( G \) are power laws, therefore \( I_n = \int \sigma^n dydz \ (n \geq 1) \) and \( H_m = \int \xi \sigma^m dydz \ (m \geq 0) \). Among them, we will focus on the following particular quantities (lowest values of \( n \) and \( m \)):

\[
E = \frac{1}{2V} \int \xi \psi rdrdzd\theta + \frac{1}{4V} \int 2\sigma^2 \frac{r^2}{r^2} rdrdz ("kinetic energy") \tag{3.42}
\]

\[
I_1 = \frac{1}{V} \int \sigma rdrdzd\theta = \frac{2\pi}{V} \int u_\theta r^2 drdz ("angular momentum"), \tag{3.43}
\]
Numerical methods

\[ I_2 = \frac{1}{V} \int \sigma^2 r dr dz d\theta = \frac{2\pi}{V} \int u_\theta^2 r^3 dr dz, \]  

(3.44)

\[ H_0 = \frac{1}{V} \int \xi r dr dz d\theta = \frac{2\pi}{V} \int \frac{\omega_\theta}{r} r dr dz (\text{"circulation"}), \]  

(3.45)

\[ H_1 = \frac{1}{V} \int \xi \sigma r dr dz d\theta = \frac{2\pi}{V} \int \omega_\theta u_\theta r dr dz (\text{"helicity"}), \]  

(3.46)

where \( V = \pi R^2 L \) is the volume of the cylinder.

We derive and give here the expressions of these quantities in terms of CK modes. And we introduce firstly that the notation

\[ \sum_{n=0,q}^{\pm \lambda} Q(n,q) = \sum_{n=0,q=1,\ldots, q_{\text{max}}}^{\pm \lambda} Q^+(n,q) + \sum_{n=0,q=1,\ldots, q_{\text{max}}}^{\pm \lambda} Q^-(n,q), \]  

(3.47)

where \( Q \) is any function depends on index \((n,q)\), and \( Q^{\pm \lambda} \) is defined in the following. Tracing back to the section of \( \lambda_{nq} \) determination, we showed that both signs of \( \pm \lambda_{nq} \) are needed. This means that as soon as the absolute value of \( \lambda_{nq} \) is determined with the equation (3.32), we must have the eigenfunctions \( A^{\pm \lambda}_{nq} \) which means the \( \lambda_{nq} \) in them are positive, and the eigenfunctions \( A^{-\lambda}_{nq} \) which have the negative \( \lambda_{nq} \). All together it can be proven that they are orthonormal basis. Hence there are also two types of coefficients \( \tilde{\xi}^v_{nq}(+\lambda_{nq}) \) and \( \tilde{\xi}^v_{nq}(-\lambda_{nq}) \) associated with these basis. Finally the notation \( Q^{\pm \lambda} \) indicates the functions calculated with \( \tilde{\xi}^v_{nq}(\pm \lambda_{nq}) \) and \( \pm \lambda_{nq} \). For the simplicity of formula, here we use the notation \( \sum_{n=0,q}^{\pm \lambda} Q(n,q) \) to represent the summation of these two types of functions depended on \( \pm \lambda_{nq} \).

Here the energy is,

\[ E = \frac{1}{2V} \int \xi \psi r dr dz d\theta + \frac{1}{4V} \int \frac{2\sigma^2}{r^2} r dr dz \]

\[ = \sum_{n=0,q}^{\pm \lambda} \frac{1}{2} \tilde{\xi}^v_{nq} \tilde{\psi}^v_{nq} + \sum_{n>0,q}^{\pm \lambda} \tilde{\xi}^v_{nq} \tilde{\psi}^v_{nq}, \]  

(3.48)
3.1 The spectral method

The angular momentum,

$$I_1 = \frac{1}{V} \int \sigma r dr dz d\theta = \frac{2 \pi}{V} \int u_{\theta} r^2 dr dz$$

$$= \frac{2 \pi}{V} \sum_{n=0,q} \xi_{nq}^v I_{nq}^{-\frac{1}{2}} \lambda_{nq} L_z \int_{0}^{\alpha} \gamma_{nq}^2 J_1(\gamma_{nq} r) dr$$

$$= \frac{2 \pi}{V} \sum_{n=0,q} \xi_{0q}^v I_{0q}^{-\frac{1}{2}} \lambda_{0q} L_z \frac{1}{\gamma_{0q}^2} \int_{0}^{a} (\gamma_{0q} r)^2 J_1(\gamma_{0q} r) d(\gamma_{0q} r)$$

$$= \frac{2 \pi}{V} \sum_{n=0,q} \xi_{0q}^v I_{0q}^{-\frac{1}{2}} \lambda_{0q} L_z a^2 J_2(\gamma_{0q} a)$$

$$= \sum_{n=0,q} \xi_{nq}^v \text{Sign}(\lambda_{nq}) \frac{\sqrt{2} J_2(\gamma_{nq} a)}{\gamma_{nq} |J_0(\gamma_{nq} a)|}.$$
The variance of the angular momentum is given by

\[
I_2 = \frac{1}{V} \int \sigma^2 r d\theta = \frac{2\pi}{V} \int u_0^2 r^3 d\theta
\]

\[
= \frac{2\pi}{V} \int \left[ \sum \xi_{nq} \xi_{nq}^* \frac{1}{2} \lambda_{nq} \gamma_{nq} J_1(\gamma_{nq} r) e^{-ik_{nq} z} \right] \left[ \sum \xi_{nq}^* \xi_{nq} \frac{1}{2} \lambda_{nq} \gamma_{nq} J_1(\gamma_{nq} r) e^{-ik_{nq} z} \right] d\theta
\]

\[
= \frac{2\pi}{V} \sum_{n,q} \xi_{nq}^* \xi_{nq} I_{nq}^{-1} \lambda_{nq} \gamma_{nq}^2 \int_0^a r^3 J_1(\gamma_{nq} r) J_1(\gamma_{nq} r) d\theta
\]

\[
= \frac{2\pi}{V} \sum_{n,q} \xi_{nq}^* \xi_{nq} I_{nq}^{-1} \frac{1}{2} \lambda_{nq} \gamma_{nq}^2 L_z \int_0^a \gamma_{nq}^3 r^3 J_1^2(\gamma_{nq} r) d\theta
\]

\[
= \frac{2\pi}{V} \sum_{n,q} \xi_{nq}^* \xi_{nq} I_{nq}^{-1} \frac{1}{2} \lambda_{nq} \gamma_{nq}^2 L_z \int_0^a \gamma_{nq}^3 r^3 J_1^2(\gamma_{nq} r) d\theta
\]

\[
\left( \frac{\gamma_{nq} a}{6} \right)^4 J_0^2(\gamma_{nq} a) \frac{2(\gamma_{nq} a)^3}{3} J_0(\gamma_{nq} a) J_1(\gamma_{nq} a) + \frac{\gamma_{nq} a^4}{6} + \frac{2\gamma_{nq} a^2}{3} J_1^2(\gamma_{nq} a)
\]

\[
\left\{ \frac{2a^2}{(\gamma_{nq1}^2 - \gamma_{nq2}^2)^3} \left[ 2\gamma_{nq1} \gamma_{nq2} J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a) + (\gamma_{nq1}^2 + \gamma_{nq2}^2) J_1(\gamma_{nq1} a) J_1(\gamma_{nq2} a) \right]
\]

\[
+ \frac{8\gamma_{nq1} \gamma_{nq2} a}{(\gamma_{nq1}^2 - \gamma_{nq2}^2)^3} \left[ \gamma_{nq2} J_0(\gamma_{nq1} a) J_1(\gamma_{nq2} a) - \gamma_{nq1} J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a) \right]
\]

\[
- \frac{a^3}{(\gamma_{nq1}^2 - \gamma_{nq2}^2)^3} \left[ \gamma_{nq1} J_0(\gamma_{nq1} a) J_1(\gamma_{nq2} a) - \gamma_{nq2} J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a) \right]
\}

\[
= \frac{2\pi}{V} \sum_{n,q} \xi_{nq}^* \xi_{nq} I_{nq}^{-1} \frac{1}{2} \lambda_{nq} \gamma_{nq}^2 L_z \int_0^a \gamma_{nq}^3 r^3 J_1^2(\gamma_{nq} r) d\theta
\]

\[
+ \frac{2\pi}{V} \sum_{n,q} \xi_{nq}^* \xi_{nq} I_{nq}^{-1} \frac{1}{2} \lambda_{nq} \gamma_{nq}^2 L_z \int_0^a \gamma_{nq}^3 r^3 J_1^2(\gamma_{nq} r) d\theta
\]

\[
= \sum_{n=0,q=1}^{\pm \lambda} \xi_{nq}^* \xi_{nq} a^2 \frac{\gamma_{nq}^2}{6} + \sum_{n=0,q=1}^{\pm \lambda} \xi_{nq}^* \xi_{nq} \frac{a^2}{3}
\]

\[
+ \sum_{n=0,q=1}^{\pm \lambda} \xi_{nq}^* \xi_{nq} \frac{\gamma_{nq}^2}{2} Sign(\gamma_{nq1} \gamma_{nq2}) \frac{4\gamma_{nq1} \gamma_{nq2} J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a)}{(\gamma_{nq1}^2 - \gamma_{nq2}^2)^2} |J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a)|
\]

\[
+ \sum_{n=0,q=1}^{\pm \lambda} \xi_{nq}^* \xi_{nq} \frac{\gamma_{nq}^2}{2} Sign(\gamma_{nq1} \gamma_{nq2}) \frac{8\gamma_{nq1} \gamma_{nq2} J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a)}{(\gamma_{nq1}^2 - \gamma_{nq2}^2)^2} |J_0(\gamma_{nq1} a) J_0(\gamma_{nq2} a)|.
\]
And the circulation,

\[ H_0 = \frac{1}{V} \int \xi r dr dz d\theta = \frac{2\pi}{V} \int \frac{\omega_\theta}{r} r dr dz \]

\[ = \frac{2\pi}{V} \sum_{n=0,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \int_0^a \frac{d}{dr} \{ -J_0(\gamma_{nq} r) \} dr \]

\[ = \frac{2\pi}{V} \sum_{n=0,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z [J_0(0) - J_0(\gamma_{nq} a)] \]

\[ = \sum_{n=0,q} \xi_{nq} \sqrt{2} [J_0(0) - J_0(\gamma_{nq} a)] \frac{\lambda_q}{a^2} J_0^2(\gamma_{nq} a) \]  \hspace{1cm} (3.51)

Finally, the helicity can be expressed by,

\[ H_1 = \frac{1}{V} \int \xi \sigma r dr dz d\theta = \frac{2\pi}{V} \int \omega_\theta u_\theta r dr dz \]

\[ = \frac{2\pi}{V} \int \left[ \sum_{n,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \int_0^a r J_1^2(\gamma_{nq} r) dr \right] \ni \frac{1}{2} \left( \sum_{n,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \int_0^a r J_1^2(\gamma_{nq} r) dr \right) \]

\[ = \frac{2\pi}{V} \sum_{n,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \int_0^a r J_1^2(\gamma_{nq} r) dr \]

\[ + \frac{2\pi}{V} \sum_{n,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \int_0^a r J_1^2(\gamma_{nq} r) dr \]

\[ = \sum_{n,q} \xi_{nq} \frac{1}{2} \lambda_q \lambda_{nq} L_z \left( \frac{1}{2} a^2 J_1^2(\gamma_{nq} a) \right) \]  \hspace{1cm} (3.52)

These expressions allow a detailed understanding of the contribution of the different modes to the value of the invariants. For expression (3.49) shows that the angular momentum is entirely contained in the modes with \( n = 0 \).
3.2 Numerical implementation

3.2.1 Time integration scheme

The time integration scheme that we are using is the classical fourth-order Runge-Kutta formula (RK4) [85]. The discretized axisymmetric control equations (3.38), fit a general form

$$\frac{dy_i(t)}{dt} = f_i(t, y_1, \cdots, y_N), \quad i = 1, \cdots, M,$$

(3.53)

where $y_i(t)$ are the time depending functions to be integrated, $M$ the total number of discretized differential equations, and $f_i$ is the right-hand side of equations (3.38) which is already known. These differential equations are solved explicitly, so that the $y_i(t_{m+1})$ at time step $t_{m+1}$ is calculated by the $y_i(t_m)$ at the previous time step $t_m$. In the RK4 method, this corresponds to

$$k_1 = hf_i(t_m, y_i(t_m)),
$$

$$k_2 = hf_i(t_m + \frac{h}{2}, y_i(t_m) + \frac{k_1}{2}),
$$

$$k_3 = hf_i(t_m + \frac{h}{2}, y_i(t_m) + \frac{k_2}{2}),
$$

$$k_4 = hf_i(t_m + h, y_i(t_m) + k_3),
$$

$$y_i(t_{m+1}) = y_i(t_m) + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5),
$$

(3.54)

where $h$ is the time step between $t_{m+1}$ and $t_m$.

3.2.2 The CFL condition

In order to choose the time step $h$, here we are applying the criteria generally referred to as the Courant-Friedrichs-Lewy (CFL) condition. The time step is calculated instantaneously as

$$h = \frac{Cdx}{U},
$$

(3.55)

where $C$ is the dimensionless Courant number, $dx$ is the smallest length interval in the discretization that is calculated by the reciprocal maximum wavenumber $1/\sqrt{(2\pi n_{max}/Lz)^2 + y_{max}^2}$, and $U$ is the root mean square of the velocity fluctuations, calculated as $U = \sqrt{2E(t_m)/3}$, where $E$ is the instantaneous total energy. Here we have conducted a comparison of many test computations with different $C$ values, and we have determined that the optimal value of $C$ is $C = 0.05$ since all the results of test computations converged with $C \leq 0.05$. 
3.2.3 Gaussian quadrature

During the time integration scheme, the integral of the non-linear term shown in equation (3.39) needs to be calculated with a good precision. Here we applied the Gaussian quadrature method [85] to achieve this goal. Without going into the theory behind this method, the formula of integration we used here is

\[
\int_a^b W(x)f(x)dx \approx \sum_{j=1}^{N} w_j f(x_j) \tag{3.56}
\]

where \( f(x) \) is the function to integrate, \((a, b)\) is the integral interval, \( W(x) \) is some known function to be chosen, \( w_j \) is the set of weights and \( x_j \) is the set of abscissas, \( N \) is the total number of this set.

There are many choices for the formula of \( W(x) \) and \( w_j \). We used the Gauss-Legendre case where

\[
W(x) = 1
\]

\[
w_j = \frac{2}{(1-x_j^2)[P_N'(x_j)]^2} \tag{3.58}
\]

in which \( P_N \) are the polynomials such that

\[
(j+1)P_{j+1} = (2j+1)xP_j - jP_{j-1}, j = 0, 1, 2, \ldots
\]

\[
P_{-1} = 0,
P_0 = 1,
\]

and the derivative \( P_N' \) is calculated with Newton’s method. This integral method appears to work reasonably well according to the results of validation described in the following.

3.3 Validation and optimization

3.3.1 Validation

Since no results on strictly axisymmetric turbulence are available in the literature, we do not have many available benchmark tests at our disposal. Our code will be therefore validated only by checking the conservation of the first inviscid invariants \( E, I_1, I_2, H_0, H_1 \) (see Subsection 3.1.5) in the case of zero viscosity. In spectral methods, the viscous term acts trivially and its implementation does not require very sophisticated tests: we have nevertheless
performed such tests, not shown here. Before doing so, we first check that these quantities are supposed to be conserved by the Euler equation with the boundary conditions considered in our study.

Demonstration of the conservation laws with our boundary conditions

Since our computation are not conducted in a closed box as in the former theoretical contributions [20, 24], here we would like to review the mathematical demonstration of the conservation of the Casimirs and general helicities ($I_n, H_m$) in axisymmetric inviscid flows, and to check if it is satisfied with our boundary conditions.

The axisymmetric Euler equations can be represented as [20, 24]

\[ \partial_t \sigma + \{ \psi, \sigma \} = 0 \]
\[ \partial_t \xi + \{ \psi, \xi \} = \partial_z \left( \frac{\sigma^2}{4y^2} \right) \]  

(3.60)

where $y = r^2/2$, $\sigma = ru_\theta$ is the angular momentum, $\xi$ is related to the azimuthal component of the vorticity by $\xi = \omega_\theta / r$ and $\psi$ is the stream function defined as $u_r = -\partial_z \psi / r$, $u_z = \partial_r \psi / r$. The following relation between $\xi$ and $\psi$ holds:

\[ \Delta_x \psi \equiv \frac{1}{2y} \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi, \]  

(3.61)

Here variables were changed from $(r,z)$ to $(y,z)$, and $\{,\}$ is defined as the Poisson bracket $\{ \psi, \phi \} = \partial_y \psi \partial_z \phi - \partial_z \psi \partial_y \phi$.

To derive the conservation laws, we will need to use the general identity

\[ \int \chi \{ \phi, \psi \} dydz = - \int \phi \{ \chi, \psi \} dydz \]  

(3.62)
which holds if one of the two fields $\chi$ or $\phi$ vanishes on the boundary of the domain. The demonstration is as follows:

\[
\int \chi \{ \phi, \psi \} dydz \\
= \int \chi (\partial_y \phi \partial_z \psi - \partial_z \phi \partial_y \psi) dydz \\
= \int \chi \partial_y \phi \partial_z \psi dydz - \int \chi \partial_z \phi \partial_y \psi dydz \\
= \int [\partial_y (\chi \phi) - \phi \partial_y \chi] \partial_z \psi dydz - \int [\partial_z (\chi \phi) - \phi \partial_z \chi] \partial_y \psi dydz \\
= \int \partial_y (\chi \phi) \partial_z \psi dydz - \int \partial_z (\chi \phi) \partial_y \psi dydz - \int \phi \{ \chi, \psi \} dydz \\
= \int [\chi \partial_z \psi]_{y_1}^{y_2} dz - \int \chi \partial_y \psi dydz - \int [\chi \partial_y \psi]_{z_1}^{z_2} dy + \int \chi \partial_z \psi dydz - \int \phi \{ \chi, \psi \} dydz.
\]

(3.63)

Supposing $\chi$ or $\phi$ vanishes on the boundary of the domain, the identity is proven.

There is another identity we need in our derivations:

\[
\{ R(\sigma), \xi \} = R'(\sigma) \{ \sigma, \xi \} = \{ \sigma, \xi R'(\sigma) \}.
\]  

(3.64)

It is a general identity that doesn’t depend on the boundary conditions:

\[
\{ R(\sigma), \xi \} = \partial_y R(\sigma) \partial_z \xi - \partial_z R(\sigma) \partial_y \xi \\
= \partial_\sigma R(\sigma) \partial_y \sigma \partial_z \xi - \partial_\sigma R(\sigma) \partial_z \sigma \partial_y \xi \\
= R'(\sigma) \{ \sigma, \xi \}
\]

(3.65)

\[
\{ \sigma, \xi R'(\sigma) \} = \partial_y \sigma \partial_z [\xi \partial_\sigma R(\sigma)] - \partial_z \sigma \partial_y [\xi \partial_\sigma R(\sigma)] \\
= \partial_y \sigma [\partial_\xi \sigma \partial_\sigma R(\sigma) + \xi \partial_\xi \sigma \partial_\sigma R(\sigma)] - \partial_y \sigma [\partial_\xi \xi \partial_\sigma R(\sigma) + \xi \partial_\xi \xi \partial_\sigma R(\sigma)] \\
= R'(\sigma) \{ \sigma, \xi \}
\]

(3.66)
Firstly, we show the conservation of energy $E$:

$$
\dot{E} = \int \frac{1}{2} \frac{\partial (\xi \psi)}{\partial t} dydz + \int \frac{\partial}{\partial t} \left( \frac{\sigma^2}{4y} \right) \frac{\partial \sigma}{\partial t} dydz
$$

$$
= \int \frac{1}{2} \frac{\partial (\xi \psi)}{\partial t} \partial \xi \partial \sigma \psi dydz + \int \frac{\partial}{\partial t} \left( \frac{\sigma^2}{4y} \right) \frac{\partial \sigma}{\partial t} dydz
$$

$$
= \int \psi \frac{\partial \xi \sigma}{\partial t} dydz + \int \frac{\sigma}{2y} \frac{\partial \sigma}{\partial t} dydz
$$

$$
= \int \left( -\psi \left\{ \psi, \xi \right\} \right) dydz
$$

$$
= \int \left( -\xi \left\{ \psi, \psi \right\} \right) dydz
$$

$$
= 0,
$$

(3.67)

Here the last two lines come from the relation (3.62), proven as above.

We now demonstrate the conservation of the Casimirs $I_n$:

$$
\dot{I}_n = n \int \sigma^{n-1} \frac{\partial \sigma}{\partial t} dydz
$$

$$
= -n \int \sigma^{n-1} \left\{ \psi, \sigma \right\} dydz
$$

$$
= -n \int \left[ \sigma^{n-1} \psi \partial_\sigma \sigma \right]^y_{y_1} dyz + n \int \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^z_{z_1} dy + n \int \psi \left( \sigma^{n-1} \left\{ \sigma \right\} \right) dydz
$$

$$
= -n \int \left[ \sigma^{n-1} \psi \partial_\sigma \sigma \right]^y_{y_1} dyz + n \int \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^z_{z_1} dy + n \int \psi (n-1) \sigma^{n-2} \left( \partial_y \sigma \partial_z \sigma - \partial_z \sigma \partial_y \sigma \right)
$$

$$
= -n \int \left[ \sigma^{n-1} \psi \partial_\sigma \sigma \right]^y_{y_1} dyz + n \int \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^z_{z_1} dy
$$

(3.68)

where in our case $y_2 = a^2/2$ corresponds to the boundary, $y_1 = 0$ corresponds to the axis, $z_2 = Lz$ and $z_1 = 0$. By the non-penetration boundary conditions we discussed previously and $J_1(0) = J_1(\gamma_{in} a) = 0$, we deduce that $\psi(y_2) = \psi(y_1) = 0$, $\sigma(y_2) = \sigma(y_1) = 0$ and $\partial_z \sigma(y_2) = \partial_z \sigma(y_1) = 0$. Also the periodic condition in the axial direction leads to $\left[ \sigma^{n-1} \psi \partial_y \sigma \right]^y_{y_1} = \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^z_{z_1}$. Thus the integral $\int \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^y_{y_1} dyz = 0$ and $\int \left[ \sigma^{n-1} \psi \partial_y \sigma \right]^z_{z_1} dy = 0$, which leads to the conservation of $\dot{I}_n = 0$. 
We now show the conservation of the generalized helicities $H_m$:

$$
\dot{H}_n = \int \left\{ F(\sigma) \frac{\partial \xi}{\partial t} + \xi F'(\sigma) \frac{\partial \sigma}{\partial t} \right\} dydz
$$

$$
= - \int F(\sigma) \left\{ \{ \psi, \xi \} \right\} dydz - \int \xi F'(\sigma) \{ \psi, \sigma \} dydz
$$

$$
= \int [F(\sigma) \xi \partial_z \psi]_{y_1}^{y_2} dy - \int [F(\sigma) \xi \partial_y \psi]_{z_1}^{z_2} dy - \int \xi \{ F(\sigma), \psi \} dydz
$$

$$
= \int \left[ F(\sigma) \frac{\sigma}{2y} \partial_z \sigma \right]_{y_1}^{y_2} dy + \int \left[ F(\sigma) \frac{\sigma}{2y} \partial_y \sigma \right]_{z_1}^{z_2} dy + \int \frac{\sigma}{2y} F'(\sigma) \{ \sigma, \sigma \} dydz
$$

$$
= \int [F(\sigma) \xi \partial_z \psi]_{y_1}^{y_2} dy - \int [F(\sigma) \xi \partial_y \psi]_{z_1}^{z_2} dy
$$

$$
= \int \left[ F(\sigma) \frac{\sigma}{2y} \partial_z \sigma \right]_{y_1}^{y_2} dy + \int \left[ F(\sigma) \frac{\sigma}{2y} \partial_y \sigma \right]_{z_1}^{z_2} dy
$$

(3.69)

after the calculation, we have found that our boundary conditions of computational domain and $J_1(0) = J_1(\gamma_{na}a) = 0$ can lead to $[F(\sigma) \xi \partial_z \psi]_{y_2} = [F(\sigma) \xi \partial_z \psi]_{y_1} = 0$, $F(\sigma) \frac{\sigma}{2y} \partial_z \sigma \]_{y_1} = 0$. By the periodic condition in the axial direction, we have $[F(\sigma) \xi \partial_y \psi]_{z_2} = [F(\sigma) \xi \partial_y \psi]_{z_1}$ and $[F(\sigma) \frac{\sigma}{2y} \partial_y \sigma \]_{z_2} = [F(\sigma) \frac{\sigma}{2y} \partial_y \sigma \]_{z_1}$. With these relations, the integrals above are zero, and the generalized helicities are conserved.

In all, it seems by our method, $I_n = 0$ and $H_n = 0$, which is consistent with the demonstration in former theoretical contributions.

**Validation of the code with invariants**

We have reviewed and checked that the quantities $E$, $I_n$ and $H_m$ were inviscid invariants with the boundary conditions that we consider. We now use these conservation laws as benchmark tests to validate our code. For this, we have conducted a series of computations with zero viscosity in order to see if the conservation of invariants was satisfied. Without any viscosity, our capacity of resolution is limited by the maximum truncated wavenumber. Thus for certain
Numerical methods

invariants we expect the conservation to hold only at the beginning of the simulation, and to be interrupted after the energy cascade has reached the smallest resolved scales.

Although there is an infinite number of inviscid invariants in axisymmetric turbulence, we decided to focus on the conservations of the quantities mentioned in Subsection 3.1.5: the total energy $E$, the angular momentum $I_1$, the Casimir invariant of second order $I_2$, the circulation $H_0$ and the helicity $H_1$. Figure 3.3 shows the time evolution of these quantities for $v = 0$, in a domain $L_z = 2\pi$ and $R = 2\pi$, with different resolutions. The quantities are all normalized by their initial values.

It can be noticed that $E$ and $H_1$ are very well conserved: $dE/dt$ and $dH_1/dt$ are always $< 10^{-9}$. This could have been expected, since the conservation of $E$ and $H_1$ with our numerical method is trivial. This can be checked from the discretized evolution equations for $E, H_1$ obtained from the NS equations. We find that the nonlinear term associated to the evolution of $E$ and $H_1$ is zero by analytical calculations. The other quantities $I_1, I_2$ and $H_0$ are also well conserved at the beginning of the computations. Their conservation is not satisfied at later times, probably when the effect of truncation becomes important: as expected, $I_1, I_2$ and $H_0$ are satisfactorily conserved during increasing times at increasing resolution. However, we have not been able to determine cascade time by simple dimensional analysis.

In order to normalize the time of flow evolvement, here we just defined a characteristic time $\tau$, which is the time scale of the initial conditions

$$\tau = \frac{l}{U},$$

where $U$ is the root mean square of the total initial energy $\sqrt{2E/3}$, and $l$ is the characteristic length scale of the initial condition

$$l \equiv \frac{\sum_{n,q} \left( \sqrt{\left( \frac{2\pi n}{L_z} \right)^2 + r_q^2} \right)^{-1} E(n,q)}{\sum_{n,q} E(n,q)}.$$

The $\tau$ in Figures. 3.3 is calculated by the specific initial condition of the testing simulation.

In all, we conclude that the inviscid invariants are reasonably well conserved by our code.

3.3.2 Resolution tests

In the last subsection we have validated the code by checking the conservation of the invariants in the inviscid case. Before presenting the results of our calculations, it is necessary to explain how to determine the required resolution. As shown in Figure 3.3, the time evolution for the
3.3 Validation and optimization

(a) Total energy

(b) Helicity

(c) Angular momentum

(d) Casimir second order

(e) Circulation

Fig. 3.3 Time evolution of the invariants in inviscid calculations.
circulation provides a more rigid constraint on the resolution needed to accurately compute the Euler equations than the other invariants. In flows with finite viscosity, one way to determine a sufficient resolution for a given Reynolds number is to run several computations of the same flow with increasing resolutions and to observe the point of convergence where all the results are identical. This is a rigorous method, but its shortcoming is the huge burden of calculation time that it requires (especially since this method must be repeated for each initial condition and each set of parameters considered). With limited computational resources, we decided to use a different method, inspired by standard practice in pseudo-spectral DNS of isotropic turbulence.

According to the Kolmogorov theory, we can estimate the dissipative scale of a turbulent flow as

\[ k_E = C_E \varepsilon_k^{\frac{1}{2}} \nu^{-\frac{3}{4}} \]  

(3.72)

where \( \varepsilon_k = dE/dt \) is the dissipation rate of kinematic energy, \( \nu \) is the kinematic viscosity and \( C_E \) is a constant which order of magnitude must be \( O(1) \). This expression can be derived by a dimensional analysis, assuming that the dissipative scale only depends on the fluid viscosity and of the dissipation rate. In decaying turbulence, the time signal of the instantaneous dissipative scale is expected to exhibit a peak (related to a maximum of turbulence intensity) and to decrease afterwards. Therefore we can compare this peak to the largest simulated wavenumbers: if the maximal value of \( k_E \) is near or below the maximal wavenumber simulated, then we can say that the resolution is more or less sufficient.

In addition to \( k_E \), we also deduced similarly, by dimensional analysis, the dissipative scales (expressed as wave numbers) of enstrophy \( k_\Omega \), of angular momentum \( k_{l_1} \) and of helicity \( k_{H_1} \). If one assumes that these quantities depend only on their dissipation rates \( \varepsilon_\Omega = d\Omega^2/2dt \), \( \varepsilon_{l_1} = |dL_1/dt| \), \( \varepsilon_{H_1} = |dH_1/dt| \) respectively, and on the kinematic viscosity \( \nu \), then the formula are

\[ k_\Omega = C_\Omega \varepsilon_\Omega^{\frac{1}{2}} \nu^{-\frac{3}{4}} \]
\[ k_{l_1} = C_{l_1} \varepsilon_{l_1}^{\frac{1}{2}} \nu^{-1} \]
\[ k_{H_1} = C_{H_1} \varepsilon_{H_1}^{\frac{1}{2}} \nu^{-\frac{3}{5}} \]  

(3.73)

where the constants \( C_\Omega, C_{l_1} \) and \( C_{H_1} \) are expected to be \( O(1) \). This method is the one used by [33] (note that they however considered less quantities than us).

Here we define the effective wavenumber of our method similarly to the work of Li et al. [33],

\[ k_{eff} = \sqrt{k_n^2 + y_q^2} \]  

(3.74)
3.3 Validation and optimization

The maximal wavenumber simulated is naturally the $k_{\text{eff}}$ calculated with maximal $k_n$ and maximal $\gamma_q$.

Two examples of comparisons between the maximal wave number calculated and the dissipative scales are shown in Figure 3.4. We can see that the magnitudes of the dissipative scales are all, and at any time, smaller than the maximal effective wavenumber simulated, therefore we can consider the resolutions of these two simulations to be sufficient.

Similar resolution tests have been carried out for all the computations considered in the following chapters.
Chapter 4

Freely decaying axisymmetric turbulence

4.1 Introduction

The numerical code which was described and validated in the previous section is used here to investigate freely decaying axisymmetric turbulence. A large number of simulations was carried out for flows in geometries of different aspect ratios, different Reynolds numbers, isotropic and anisotropic initial conditions, containing energy at the large and/or at the small scales. A complete investigation of all the different possible cases in the parameter space is not feasible in the context of this project. We have therefore chosen to present a small subset of the results, representative of a large number of simulations, and showing some key-features we observed in the dynamics of decaying axisymmetric turbulence.

4.2 Axisymmetric turbulence and predictions from statistical mechanics

4.2.1 Equations and invariants

The axisymmetric Navier-Stokes equations in cylindrical coordinates \((r, \theta, z)\) read for the three velocity components \((u_r, u_\theta, u_z)\),

\[
\begin{align*}
\partial_t u_r &+ u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \right) \\
\partial_t u_\theta &+ u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} \right) \\
\partial_t u_z &+ u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_z}{\partial r} \right) + \frac{\partial^2 u_z}{\partial z^2} \right),
\end{align*}
\]
Freely decaying axisymmetric turbulence

where \( p \) is the pressure and \( \nu \) the viscosity, and incompressibility is expressed by,

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_r}{\partial z} = 0. \tag{4.4}
\]

These are the equations which are considered in the following in a cylindrical domain. It was shown that the above system, for \( \nu = 0 \) could be written as a function of two quantities. First, the potential vorticity \( \xi \) associated with the azimuthal component of the vorticity is defined by

\[
\xi = \frac{1}{r} (\partial_z u_r - \partial_r u_z) = \frac{\omega_\theta}{r}. \tag{4.5}
\]

This vorticity can be computed from the streamfunction using the generalized Laplacian, \( \Delta_* \),

\[
\Delta_* \psi \equiv \frac{1}{2y} \partial_y^2 \psi + \partial_y^2 \psi = -\xi, \tag{4.6}
\]

where the variable \( y \) is defined as \( y = r^2/2 \). We further introduce the angular momentum \( \sigma = ru_\theta \). The advantage of these definitions is that the equations (4.1) reduce to

\[
\partial_t \sigma + \{ \psi, \sigma \} = 0, \tag{4.7}
\]

\[
\partial_t \xi + \{ \psi, \xi \} = \partial_z \left( \frac{\sigma^2}{4y^2} \right). \tag{4.8}
\]

and where the Poisson brackets are defined classically, as

\[
\{ \psi, \phi \} = \partial_y \psi \partial_z \phi - \partial_z \psi \partial_y \phi \tag{4.9}
\]

We have hereby defined a two-dimensional system, described by \( \psi, \sigma \), as a function of the variables \( y, z \).

In general in turbulent flows it is the invariants of the system which play an important role in the nonlinear dynamics. Invariants are integral quantities (or volume averages in the present case), which are conserved in the absence of viscosity, i.e. conserved by the Euler-equations. The invariants of this system are the energy

\[
E = \frac{1}{2} \left( \langle \xi \psi \rangle + \frac{1}{2} \langle \sigma^2 \rangle \right), \tag{4.10}
\]

all moments of the angular momentum,

\[
I_m = \langle \sigma^m \rangle, \tag{4.11}
\]
and generalized helicities, defined by

\[ H_n = \langle \xi \sigma^n \rangle. \] (4.12)

We note here a degenerate case. If in the above system \( \sigma \) tends to zero, or more generally, the RHS of (4.8), \( \partial_z \left( \sigma^2 \langle 4y^2 \rangle \right) = 0 \), the two equations are decoupled and equation (4.8) becomes a pure mixing equation for \( \xi \), identical to the 2D Euler equations [19]. In this case new invariants appear, i.e. Casimirs of the potential vorticity,

\[ \Gamma_n = \langle \xi^n \rangle. \] (4.13)

are conserved and a dynamics as predicted by the Miller-Robert-Sommeria theory should result. In this two-dimensional limit the invariants that will play the most important role are the poloidal energy, defined by

\[ E_P = \frac{1}{2} \langle \xi \psi \rangle \] (4.14)

and the potential enstrophy

\[ Z = \langle \xi^2 \rangle. \] (4.15)

We will also introduce the poloidal and toroidal enstrophy. The poloidal enstrophy is defined as

\[ \Omega_P = \langle \omega_\theta^2 \rangle \] (4.16)

and is thus computed from the toroidal component of the vorticity. But since this vorticity is associated with the velocity-gradients in the poloidal plane, we have chosen the name poloidal enstrophy. The toroidal enstrophy is associated with the other two components of the vorticity,

\[ \Omega_T = \langle \omega_r^2 + \omega_\xi^2 \rangle. \] (4.17)

### 4.2.2 Predictions of the theory

Two families of predictions are possible in the light of the foregoing considerations. First a dynamics similar to two-dimensional turbulence (governed by equation (4.8) with zero RHS), where the azimuthal component of the velocity either zero or invariant with respect to the \( z \)-direction, or a system where \( \sigma \) and \( \xi \) are coupled and the most probable state should verify the two equations (4.7) and (4.8) simultaneously.

**Fully axisymmetric turbulence.** In the case of full axisymmetric turbulence, steady solutions of the axisymmetric Euler equations (4.7) and (4.8) were shown [20] to correspond to
flow-fields obeying the following relations between $\sigma$, $\xi$ and $\psi$.

$$\sigma = F(\psi) \quad (4.18)$$

$$\xi - \frac{FF'}{r^2} = G(\psi) \quad (4.19)$$

where $F(\psi)$ and $G(\psi)$ are arbitrary functions of $\psi$. It can be shown by substituting these expressions in (4.7) and (4.8), that this yields $\partial_t \sigma = \partial_t \xi = 0$.

If equilibrium states exist, meta-stable for times shorter than the viscous diffusion time of the flow, they are thus characterized by these expressions. What the shape of these expressions is, cannot be concluded directly from these equations without further assumptions. In the spirit of investigations of two-dimensional turbulence, such assumptions can be based on phenomenological variational principles of the selective decay type.

References [20, 24] considered the Energy-Helicity-Casimir funtional to predict a particular Euler solution, at high Reynolds number. The idea is a generalization of the Arnol’d energy-Casimir functional with the constraints of invariants. The formula of the functional can be written as $A = E + \sum_m I_m + \sum_n H_n$, and if we consider the optimization problem of $A$,

$$\delta A = \delta(E + \sum_m I_m + \sum_n H_n) = 0, \quad (4.20)$$

we can obtain the equations that characterize a steady solution of the axisymmetric Euler equations. By taking variations in $\sigma$ and $\xi$, we have

$$\psi + R(\sigma) = 0, \quad (4.21)$$

$$\frac{\sigma}{2y} + D'(\sigma) + \xi R'(\sigma) = 0, \quad (4.22)$$

in which $R$ and $D$ are functions of $\sigma$. Retaining only the lower order invariants $E$, $I_1$, $I_2$, $H_0$ and $H_1$ in the definition of $A$, the minimization problem, can be written as

$$\delta H_1 + \lambda \delta I_2 + \mu \delta E + \alpha \delta I_1 + \beta \delta H_0 = 0, \quad (4.23)$$

taking the variation of $\xi$ and $\sigma$ respectively, we have

$$\sigma = -\mu \psi - \beta, \quad (4.24)$$
4.3 Numerical method and initial conditions

\[ \xi + \mu \frac{\sigma}{r^2} = -2\lambda \sigma - \alpha. \]  
(4.25)

Furthermore, by substituting equation (4.24) into equation (4.25), we obtain

\[ \xi + \mu \frac{\sigma}{r^2} = 2\lambda \mu \psi + 2\lambda \beta - \alpha. \]  
(4.26)

Two of the equations (4.24), (4.25) and (4.26) give the general formula of the critical points obtained from the variational problem (4.23). The resulting solution is also a steady solution of the axisymmetric Euler equations.

**Two-dimensional turbulence in the poloidal plane.** In the case of small, or axially invariant \( \sigma \), the nature of the flow changes to a purely two-dimensional dynamics in the poloidal plane. The variational equation depends then on the invariants of this two-dimensional system, the poloidal energy and the casimirs of the toroidal vorticity. In this case the Miller-Robert-Sommeria theory for 2D flows provides an expression based on all invariants. In freely decaying two-dimensional turbulence the solution tended to the hyperbolic-sine prediction obtained from the point-vortex model by Joyce and Montgomery [5, 6]. Focusing on the case of zero circulation, the solution is given by

\[ \xi \sim \sinh(\beta \psi), \]  
(4.27)

where the exact shape can be refined taking into account additional Casimirs. It will be shown that this quasi-two-dimensional poloidal dynamics is indeed relevant for the present system.

4.3 Numerical method and initial conditions

4.3.1 Eigenfunctions of the curl

The numerical method was described in chapter 3 of the present manuscript and we briefly recall some features here. Velocity fields are projected on a basis consisting of Chandrasekhar-Kendall eigenfunctions of the curl \( A_{nq} \), defined such that

\[ \nabla \times A_{nq} = \lambda_{nq} A_{nq}. \]  
(4.28)

where \( n, q \) are the mode-numbers corresponding to the discretization in the axial and radial direction, respectively. The functions \( A \) are an orthonormal basis of solenoidal functions,
allowing a precise, non-diffusive representation of the dynamics of the velocity field, represented by the coefficients \( \xi_{nq}(t) \),

\[
v(r, z, t) = \sum_{nq} \xi_{nq}(t) A_{nq}(r, z).
\]  

(4.29)

The method is fully spectral. Apart from the mathematical esthetics of this modal decomposition, the numerical advantage of such a method is that no aliasing-error is present, since multiplications are evaluated in spectral space using convolution products. A further advantage is that the precise modal structure is directly visible and the contributions of the different modes on the value of invariants such as angular momentum, helicity, energy etc. can be evaluated directly.

An important disadvantage is, however, that for large numbers of modes the convolution products representing the nonlinearity of the Navier-Stokes equations become prohibitively expensive. For this reason, simulations of Reynolds numbers up to \( 10^3 \) approximately will be reported in this work. Despite this limitation, it will be shown that the dynamics we can consider using this technique allow already to consider a very rich turbulent system.

The code is a modification of the one described in references [31–33] developed for 2D turbulence. The same method was used in refs [35–37] to consider 3D MHD. The Chandrasekhar-Kendall eigenfunctions of the curl \( A_{nq} \) are a combination of Fourier-modes in the z-direction and Bessel-functions in the radial direction. To characterize the lengthscale of the flow-field, we define an effective wavenumber based on the zeros \( \gamma_q \) of the Bessel-functions and the azimuthal wavenumber \( k_n \),

\[
w_{nq} = \sqrt{\left( \frac{n2\pi}{L} \right)^2 + \gamma_q^2},
\]

(4.30)

which allows to determine an integral lengthscale by the relation

\[
l = \frac{\sum_{n,q} w_{nq}^{-1} E(n, q)}{\sum_{n,q} E(n, q)}
\]

(4.31)

where \( E(n, q) \) is the energy associated with mode \( \xi_{nq} \). The integral timescale is defined by \( \tau = l/U \). The Reynolds number is defined as

\[
Re = \frac{UL}{v}
\]

(4.32)
where \( U = \sqrt{2E/3} \) and \( L \) is the height of geometry domain. The computational domain is shown in Figure 4.1, with \( R = L = 2\pi \). Time-integration is performed using a fourth-order Runge-Kutta method. Further technical details on the method can be found in references [33, 36].

The simulations all start with an initial amount of kinetic energy \( E = 1.5707 \), the viscosity is \( \nu = 0.01 \) and the resolution is \( n \times q = 40 \times 60 \) modes.

### 4.3.2 Initial conditions

Since this study reports on the first simulations of axisymmetric turbulence, the character of the investigation is necessarily exploratory and we do not claim to give an exhaustive description of all possible cases. We will rather concentrate on three sets of simulations, starting from random initial conditions with different initial values of the invariants. The first two of our simulations contained a comparable initial amount of energy in all three coordinate directions (the RMS values of \( u_r, u_\theta \) and \( u_z \) were the same order of magnitude) and a finite value of the mean helicity \( H_1 \). Initial conditions were created with either zero or non-zero values of angular momentum \( I_1 \) and circulation \( H_0 \). The third case we consider is characterized by a dominant toroidal energy \( E_T \), the poloidal energy containing only a few percent of the total energy.

In Figure 4.2 we show the energy distribution of the three initial conditions. As explained in chapter 3, the total angular momentum \( I_1 \) and total circulation \( H_0 \) depend only on modes with \( k_n = 0 \). This difference is visible in the initial conditions in Figure 4.2.

We have, among the infinite number of possible cases, chosen here to focus on cases where the initial energy distribution is centered around the large scales, i.e., low values of \( n \) and \( q \). Further studies with different initial conditions constitute a natural perspective of the present work. But since the experimental observations of axisymmetric average states in the
Von Kármán flow involve the large scales of the flow we will focus the present simulations on these initial conditions.

Table 4.1 reports all initial values of the considered integral quantities, as well as their value, at \( t = 20\tau \), in % of their value at \( t = 0 \).

4.4 Results

In this section we will discuss the results for three distinct cases. The first is the case where initially the angular momentum \( I_1 \) and the circulation \( H_0 \) are equal to zero. Furthermore, the initial toroidal and poloidal energy are comparable. In the second case the initial energy is mainly concentrated in the largest scale \((n = 0, q = 1)\) with relatively strong initial azimuthal flow and large initial value of angular momentum. The third case starts with nearly only toroidal energy, \( E_T(0) \gg E_P(0) \).
Table 4.1 Table with initial values of a certain number of integral quantities and their value at $t = 20\tau$.

<table>
<thead>
<tr>
<th></th>
<th>Run1 (t=0)</th>
<th>t=20</th>
<th>Run2 (t=0)</th>
<th>t=20</th>
<th>Run3(t=0)</th>
<th>t=20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>1.5707</td>
<td>0.3884</td>
<td>1.5707</td>
<td>0.6613</td>
<td>1.5707</td>
<td>0.6093</td>
</tr>
<tr>
<td>$H_0$</td>
<td>0.0</td>
<td>-0.0489</td>
<td>0.021</td>
<td>0.082</td>
<td>0.0329</td>
<td>0.0147</td>
</tr>
<tr>
<td>$H_1$</td>
<td>2.5535</td>
<td>0.2076</td>
<td>0.676</td>
<td>0.314</td>
<td>0.8238</td>
<td>0.3083</td>
</tr>
<tr>
<td>$I_1$</td>
<td>0.0</td>
<td>0.0163</td>
<td>3.6</td>
<td>3.29</td>
<td>4.149</td>
<td>2.5314</td>
</tr>
<tr>
<td>$I_2$</td>
<td>16.594</td>
<td>0.9468</td>
<td>28.66</td>
<td>13.31</td>
<td>83.155</td>
<td>29.955</td>
</tr>
<tr>
<td>$E_T$</td>
<td>0.6805</td>
<td>0.0248</td>
<td>1.0397</td>
<td>0.4655</td>
<td>1.5381</td>
<td>0.5593</td>
</tr>
<tr>
<td>$E_P$</td>
<td>0.8902</td>
<td>0.3636</td>
<td>0.531</td>
<td>0.1958</td>
<td>0.0326</td>
<td>0.05</td>
</tr>
<tr>
<td>$Z$</td>
<td>6.8446</td>
<td>0.1518</td>
<td>2.361</td>
<td>0.2814</td>
<td>0.0358</td>
<td>0.0892</td>
</tr>
<tr>
<td>$\Omega_P$</td>
<td>14.556</td>
<td>2.0267</td>
<td>7.507</td>
<td>1.3760</td>
<td>0.2567</td>
<td>1.161</td>
</tr>
<tr>
<td>$\Omega_T$</td>
<td>10.171</td>
<td>0.5695</td>
<td>10.323</td>
<td>1.5387</td>
<td>11.728</td>
<td>7.361</td>
</tr>
</tbody>
</table>

4.4.1 A quasi two-dimensional dynamics, when $I_1(t = 0) = 0$.

Time-evolution of the integral quantities. An important question is how the different integral quantities will evolve as a function of time. Figure (4.3) shows the time evolution of some important integral quantities in Run1 ($H_0 = I_1 = 0$). Time-evolutions are shown up to 20 initial turnover times $\tau$. Simulations were all continued until $t/\tau = 100$, but the general tendencies do not change for longer times: all quantities monotonically tend to zero for longer times. We therefore focus here on the dynamically important time-interval in which some quantities are almost completely dissipated, while others still subsist.

It is clear from these results that the most robust quantity in these simulations is the poloidal kinetic energy. Helicity $H_1$, toroidal kinetic energy $E_T$ and the variance of angular momentum fluctuations $I_2$ all three decay more rapidly. The angular momentum $I_1$ and circulation $H_0$ remain close to zero but are not exactly conserved. This seems counterintuitive, since inviscid invariants of a system are in general not created in the presence of viscosity. However it is not in disagreement with the physics of the flow. Indeed, angular momentum and circulation can be created by the combined effect of nonlinear mode coupling and viscosity as we will show now.

Modal dynamics. The generation of angular momentum can be understood by investigating the modal distribution of energy (Figure. 4.4). Only a small subset of modes, with small values of $n, q$ are shown. After a violent nonlinear phase for $t/\tau < 20$, where the energy is actively redistributed over a large number of modes, the modal activity relaxes and at long times, for $t/\tau > 50$, the mode $\xi_{01}^v$ becomes dominant. This mode contains a finite value of angular momentum. In the inviscid case, its angular momentum would be compensated by
Fig. 4.3 Time evolution of integral quantities in the simulation with initially $H_0 = I_1 = 0$. (a) Energy $E$, Helicity $H_1$ and angular momentum fluctuations $I_2$, all normalized by their initial value. (b) Evolution of the circulation $H_0$ and angular momentum $I_1$. (c) Toroidal and poloidal energy $E_T, E_P$, and enstrophy normalized by their initial value. (d) Ratio of poloidal to total energy.
4.4 Results

Fig. 4.4 (a) Time-evolution of modal energy for low wavenumber modes for the run with zero initial $I_1$ and $H_0$. (b) and (c) Energy spectrum at $t = 2.2\tau$ and at $t = 50\tau$.

opposite contributions of modes with larger $q$-values. However, in the presence of viscosity, these modes are dissipated more rapidly than the $\xi_{01}^1$ mode, resulting in the net generation of angular momentum at large times. We will return to this in Chapter 5.

In Figure. 4.4 we also show the complete $n, q$ wavenumber spectrum at two particular time-instants. The first instant $t = 2.2\tau$ corresponds to the phase of maximum modal activity, and it is observed that a large number of modes contains the energy. At $t = 50\tau$ the system approaches its equilibrium state, and the energy is centered around a few low $n, q$-number modes.

Flow-visualizations. From Figure 4.3 it can be seen that the flow tends to a quasi-two-dimensional dynamics in the poloidal plane. It is then expected that, if the kinetic energy in the poloidal plane is still sufficiently large, large structures reminiscent of the MRS theory of 2D flows should be observed in the poloidal plane. This is illustrated in Figure 4.5, where a
Fig. 4.5 Run with zero initial $I_1$. Flow visualizations. Iso-contours of stream function in the $r-z$ plane at four different times, $t = 0$ the initial field; $t = 2.2\tau$ where the turbulence activity is nearly the most violent; $t = 10\tau$ and $t = 50\tau$ where the large scale coherent structures emerge.

dipolar structure is seen to be formed as a final state, before viscosity dissipates the rest of the kinetic energy.

**Functional relations, final states.** In the quasi two-dimensional case observed, statistical mechanics predict the appearance of a functional relation between $\xi$ and $\psi$, of the form $\xi \sim \sinh(\beta \psi)$, as discussed before. This is verified in Figure. 4.6(b). It is observed that a hyperbolic sine is compatible with the scatter-plot at long times. Quantitatively, in Figure. 4.6(c) a hyperbolic sine can be fitted to the data with a correlation which increases upto 80% over the considered time-interval.

### 4.4.2 non-Zero initial angular momentum

In two-dimensional turbulence it was shown that the existence of an additional invariant can change the dynamics importantly. For instance in a number of papers [31–33] it was
4.4 Results

Fig. 4.6 Zero initial $I_1$: evolution of the correlations among $\sigma$, $\xi$ and $\psi$. 

(a) $(\sigma, \psi)$. 

(b) $(\xi, \psi)$. 

(c) Correlation coefficients of $(\xi, \sinh(0.41\psi + 0.64))$. 

Re=643 

corr $(\xi, \sinh(0.41\psi + 0.64))$
shown that the presence of a significant amount of angular momentum could change the final decaying state in a circular domain from a dipolar to a monopolar structure. Furthermore, in the framework of the Euler-equations, variational principles allowed to show how the circulation can change the equilibrium solutions [51] in a rectangular domain. In the present section we therefore investigate this for axisymmetric turbulence, by considering initial conditions containing a substantial amount of angular momentum \( I_1 \). In the previous case we considered, the initial toroidal and poloidal energy were comparable. It was shown that the toroidal energy was dissipated more rapidly, resulting in a purely poloidal dynamics. We have therefore, in addition to setting a value for the angular momentum, introduced a more dominant initial toroidal energy.

**Time-evolution of the integral quantities.** We see in Figure 4.7 that the most robust quantity is the angular momentum \( I_1 \). The toroidal and poloidal energy decay at nearly the same pace. This behavior shows that we are now in a case different from the former quasi 2D scenario, where the poloidal energy is much more conserved while the toroidal energy decays quickly to zero. The evolution of \( H_0 \) is omitted. Its initial value is small, and it increases in time, as in case 1 to a larger value, still negligible in normalized units.

We further see that the helicity \( H_1 \) is better conserved than in the previous case. Since \( H_1 \) is zero in 2D turbulence, the fact that it does not directly goes to zero also indicates that we are closer to a 3D decaying field. The way that helicity is related to the 3D character of the flow can be further illustrated as follows. In axisymmetric turbulence it can be shown that the toroidal helicity \( H_1^T = \int \omega_\theta u_\theta dV \) always equals the poloidal helicity \( H_1^P = \int (\omega_r u_r + \omega_z u_z) dV \),

\[
H_1 = H_1^T = H_1^P,
\]

hence we have an upper bounded for \( H_1 \),

\[
H_1^T = H_1^P \leq \min\{(u^T \omega^T), (u^P \omega^P)\},
\]

in which \( u^{T,P} \) and \( \omega^{T,P} \) are the root mean square of the toroidal and poloidal components of energy and enstrophy. According to this upper bound, in the zero \( I_1 \) case, when the toroidal component decays fast to zero, the field becomes a quasi 2D situation and \( H_1 \) also decay to zero. In the present case, where \( E_P \) and \( E_T \) decay together, \( H_1 \) is not constrained to decay as fast as in case 1.

**Modal dynamics.** The modal distribution of energy in this scenario is shown in Figure 4.8. We see that the energy in the largest scale \((n = 0, q = 1)\) contains always a significant
4.4 Results

Fig. 4.7 Finite initial $I_1$ and $H_0$: history of quantities which is normalized by their initial values.

portion of the whole energy, which increases with time. This mode contains both toroidal and poloidal energy, and since it dominates the total energy, the ratio of toroidal to poloidal energy remains approximately constant. The energy of the other modes decays more rapidly than the $(0, 1)$ mode, but does not strongly influence the large-scale dynamics.

In Figure. 4.8 we also show the complete $n, q$ wavenumber spectrum at two particular time-instants. The first instant $t = 3 \tau$ corresponds to the phase of maximum modal activity, and it is observed that a large number of modes contains the energy. At $t = 70 \tau$ the system still approaches its equilibrium state, and the energy starts to center around a few low $n, q$-number modes.

**Flow-visualizations.** From Figure. 4.9, we can see that a monopolar structure is forming in the free decaying axisymmetric scenario, analogical to the prediction of the statistical mechanics for the Euler equations. This time we know from the Figure. 4.7(b) that both poloidal and toroidal energy are sufficiently large, thus it is no-longer a quasi-two-dimensional dynamics only in the poloidal plane but a slowly decaying axisymmetric state. Under a large $I_1$, we have observed a formation of large-scale monopole in the stream-function field. The poloidal vorticity $\xi$ field does not contain much structure in this case

**Functional relations, final states.** As mentioned above in the equation (4.18) and (4.19), quasi-stationary states of the axisymmetric Euler equations should yield two functional relations $F(\psi)$ and $G(\psi)$. In the Figure. 4.10 we see that a linear function of $F(\psi) = \sigma$ and also a linear function of $G(\psi) = \xi - F'F/r^2$ are not excluded. However, at $t = 70 \tau$, the function of $G(\psi)$ is not as obvious as $F(\psi)$, but if we only look at the field with $r > 0.5R$, There is a line graph showing the decay of energy with time, and another line graph showing the decay of energy with time for another scenario.
Fig. 4.8 (a) Time-evolution of modal energy for low wavenumber modes for the run with non-zero initial $I_1$ and $H_0$. (b) and (c) Energy spectrum at $t = 3\tau$ and at $t = 70\tau$. 
Fig. 4.9 Run with non-zero initial $I_1$ and $H_0$. Flow visualizations. Iso-contours of stream function in the $r - z$ plane at four different times, $t = 0$ the initial field; $t = 2.5\tau$ where the turbulence activity is nearly the most violent; From $t = 25\tau$ to $t = 70\tau$ where the coherent structures begin to emerge.
Fig. 4.10 Non-zero initial $I_1$ and $H_0$: evolution of the correlations among $\sigma$, $\xi - \sigma' \sigma/r^2$ and $\psi$.

where the monopole is present (denoted ’coherent structure region’ in the plots) one observes a more clearly linear function of $G(\psi)$. It might be that the current Reynolds number is not large enough to see if a beltrami state will actually occur. Checking this requires a more efficient simulation method.

### 4.4.3 Strongly dominant toroidal energy

From the first case we considered we showed that the distinct dynamics of the toroidal and poloidal energy can greatly affect the dynamics of free decaying axisymmetric turbulence. It was observed that the poloidal energy was far better conserved than the toroidal energy. The toroidal velocity, appears directly in the coupling term on the RHS of equation (4.7). A question is now whether an initially toroidal field, with only small poloidal fluctuations, can result in a dominant poloidal field, through this coupling term.

**Time-evolution of the integral quantities.** In Figure. 4.11(a), we see that $I_1$ is still the most robust invariant, and $E$, $H_1$ and $I_2$ all decay in the similar pace because they all follow the decay of $E_T$. However, at the earlier stage of evolution, $E_P$ increases importantly, and this can only have been caused by the toroidal source term. It is shown in the Figure. 4.11(c) that the the fraction of poloidal energy increases over the whole time-interval from 2% to 16%. We conjecture that this trend should be even stronger at higher Reynolds number.

**Flow-visualizations.** Figure 4.12 shows the evolution of the field in this case. Different from the zero $I_1$ case, one observation is a violent mixing of $\sigma$. The initially almost axially
Fig. 4.11 Extreme case with only $E_T$: (a) and (b) history of quantities which is normalized by their initial values; (c) fraction of poloidal energy to toroidal energy.
Fig. 4.12 Run with absolutely dominant $E_T$. Flow visualizations. Iso-contours of stream function in the $r-z$ plane at four different times, $t = 0$ the initial field; $t = 15\tau$ where the mixing activity of $\psi$ is nearly the most violent; From $t = 28\tau$ to $t = 100\tau$ where the coherent structures begin to emerge.

invariant azimuthal velocity field does not remain so, but is perturbed by instabilities, which seem to be most active near the wall of the domain.

4.4.4 Robustness of the effects. Low Reynolds number dynamics, rectangular geometry

At the moderate Reynolds numbers that we consider viscous effects are far from negligible, and a legitimate question is whether our results are not a purely viscous dynamics. In order to discard that possibility, we have also computed a low Reynolds number case where the initial conditions are kept the same as in run1, but the initial Reynolds number is lowered by a factor 100, by increasing the viscosity by this same factor. The results are shown in Figure 4.13. The important difference of the evolution of the integral quantities is that the toroidal and poloidal energy decrease with the same rate, unlike the results of run 1 at the
higher Reynolds number simulations. The observed persistence of the poloidal energy for a more rapidly decreasing toroidal energy seems therefore to be due to the nonlinearity of the system.

Another interesting feature is that, even at this low Reynolds numbers, the angular momentum is a very robust quantity. A finite amount of angular momentum is generated by the flow during the first turn-over time of the evolution. Subsequently the angular momentum decays over a slow timescale, associated with the viscous decay of mode \((n = 0, q = 1)\), as shown in Figure 4.13(c). At even longer times (not shown) the angular momentum decays to zero value.

To further illustrate the robustness of the results we have also carried out simulations in a domain with an aspect ratio of 1.7. Qualitatively the results seemed unchanged. Flow visualizations are shown in Figure 4.14.
Fig. 4.14 Flow visualizations of simulations carried out in a domain of size $R = \pi, L = 1.7\pi$. 
4.5 Discussion

4.5.1 Case 1: Two-and-a-half dimensional turbulence

The first case we considered in this investigation, with small initial angular momentum, finds its analogy in the case of quasi-static magnetohydrodynamics (QS-MHD). We find it interesting to draw a parallel with such a dynamics, characteristic of liquid metals in the presence of a strong imposed magnetic field, which are well approximated by quasi-static magnetohydrodynamics, where the influence of the magnetic field on the flow is represented by an anisotropic damping term. After a linear phase, in which the anisotropy is developed [86], the dynamics are well represented by two-and-a-half dimensional turbulence [87], invariant along the direction of the magnetic field, and where the component of the velocity field aligned with the magnetic field $u_\parallel$ is advected as a passive scalar, by the velocity field perpendicular $u_\perp$ to the magnetic field. In this case the dynamics of the two fields, $u_\parallel$ and $u_\perp$ become independent, given by the equations

$$\partial_t u_\parallel + \{\psi_\perp, u_\parallel\} = 0,$$

$$\partial_t \omega_\perp + \{\psi_\perp, \omega_\perp\} = 0,$$

where $\Delta \psi_\perp = -\omega_\perp$ and $\psi_\perp$ is the streamfunction associated with $u_\perp$. This system is equivalent to the present case of axisymmetric turbulence, if the RHS term of equation (4.8) is neglected. In the MHD flow, the dynamics of the parallel flow can then be well described by a direct cascade process, where the variance of $u_\parallel$ rapidly decays [88] and the perpendicular dynamics is characterized by a long-lasting two-dimensional turbulent state (if the viscosity is sufficiently large). The inviscid dynamics are then described by the same equations as (4.7), with zero right-hand-sides. It is therefore not surprising that in the present case we observe at long times a two-dimensional dynamics in the poloidal plane, if the initial value of the angular momentum fluctuations $\sigma$ is weak.

4.5.2 Case 2: unimodal dynamics

The second case, in which the $(n = 0, q = 1)$ mode was initially strong, showed that this same mode remained dominant during the whole dynamics. The difference with the first case is striking, since this mode is three-dimensional. No quasi-2D flow-state was attained. The flow-visualizations showed that the long-time pattern of the flow contained a monopolar structure, different from the dipolar structure in case 1.

The situation is quite similar to the work of Li and Mongomery [31, 33], where it was shown in two-dimensional turbulence in a circular domain (where the angular momentum is...
also an invariant of the inviscid system), that the final, long-lasting state could drastically change if the initial conditions contained, or not, a significant amount of angular momentum.

It is interesting that a monopolar structure is observed, and not an axially invariant solution, since the dominant mode is axially invariant \( n = 0 \). The other modes have thus a non-trivial contribution to the flow, it seems, and the \((n = 0, q = 1)\) mode does not fully determine the flow topology.

Some, rather scarce, agreement with theoretical predictions was observed in the scatter plots of the different quantities, but this needs definitely better resolved simulations for confirmation.

### 4.5.3 Case 3: very strong azimuthal flow

The third case showed that the mechanism observed in the first case is rather robust. Indeed, when initially more than 97% of the energy is contained in the toroidal component, does the poloidal energy still dominate at large times? This does seem to be a possibility if the Reynolds number is large enough initially. Already at the modest Reynolds number we consider the poloidal energy increased, proportionally, from 2% to 16% over the time-interval \( 0 < t < 100\tau \). This shows that the axisymmetric system, in the absence of forcing terms, tends to a quasi-two dimensional dynamics in the poloidal plane.

### 4.6 Conclusion

An obvious question at this point is how relevant these ideas are for the interpretation of the experimental results in the Von Kármán flow. More precisely, can we conclude now that the agreement of statistical mechanics with the observations on the mean-flow profiles is fortuitous? Or can we on the other hand confirm these theoretical speculations?

Such conclusions are difficult, if not impossible, at the moment. One of the reasons is that the Von Kármán flow is not periodic in the axial direction, since the impellers continuously break the \( z \)-invariance. Furthermore, in the experiment a certain mean helicity is somehow imposed and should therefore appear as a solid constraint on the dynamics. However, it is not a constraint which stems from the equations and their invariants, but a constraint imposed by the geometry of the experiment. It seems from the present results that the helicity is not a robust constraint of the dynamics if it is not enforced through the boundary conditions.

What we have learned, however, is that axisymmetric turbulence seems to form large coherent structures, angular momentum is a very robust invariant, and azimuthal fluctuations
decay more rapidly than poloidal ones. To better understand the underlying mechanisms, we will focus in the next chapter on cascades of energy and other invariants.
Chapter 5

Spectral analysis of the flow

A common way of characterizing the multiscale statistics of turbulence consists in using a spectral analysis. As recalled in Chapter 2, in the inertial range of forced isotropic incompressible turbulence, the 2D turbulence exhibits an inverse cascade of energy compatible with $E(k) \sim k^{-5/3}$ and a direct cascade of enstrophy compatible with the scaling $E(k) \sim k^{-3}$ [65]. In 3D turbulence, the classical behavior of the direct cascade of energy is compatible with $E(k) \sim k^{-5/3}$. No results are currently available for strictly axisymmetric turbulence, but a dual cascade was recently evidenced through the spectral analysis of the von Kármán flow (whose mean is axisymmetric) [38]. Spatial power spectra of the velocity fluctuations were measured, for the first time in this flow without using the Taylor hypothesis. As the mean flow was observed to be Beltrami, the helicity had a well defined sign and the equality $H_1(k) = kE(k)$ holds. Based on this assumption, the authors hypothesized that helicity dominates the direct cascade (largest $k$), and that energy dominates the inverse cascade (smallest $k$). This scenario is compatible with the slopes of the measured energy spectra: the inverse cascade is characterized by $E(k) \sim k^{-1}$ (a scaling compatible with a nonlocal energy cascade), and the direct cascade by $k^{-2} \sim k^{-7/3}$ (local or nonlocal helicity cascade), depending on the Reynolds number.

We perform in the present chapter a spectral analysis of strictly axisymmetric turbulence. Traditionally, the features of energy spectra in turbulence can be explained by the existence of cascades of the inviscid global invariants of the Navier-Stokes equations. We will therefore investigate possible cascades of the first invariants of the axisymmetric Euler equations, i.e., energy $E^{tot}$, helicity $H_1^{tot}$, angular momentum $I_1^{tot}$ and circulation $H_0^{tot}$ (in this chapter we will denote the total quantities with a superscript $tot$, to distinguish them from their spectral density counterparts).

In particular, the fact, evidenced in Chapter 4, that helicity decays much faster than energy is still unexplained. Assuming that these two quantities effectively cascade in axisymmetric
Spectral analysis of the flow (which might be possible for highly helical flows), this may be due to the different natures of the helicity and energy cascades. Indeed, the time evolution equations of the modal energy and helicity, $E(n,q)$ and $H_1(n,q)$, obtained by the discretized Navier-Stokes equations, formally read:

$$\frac{dE(n,q)}{dt} = T_E(n,q) - 2\nu\lambda_{nq}^2E(n,q), \quad (5.1)$$

$$\frac{dH_1(n,q)}{dt} = T_{H_1}(n,q) - 2\nu\lambda_{nq}^2H_1(n,q), \quad (5.2)$$

where $T_E(n,q)$ and $T_{H_1}(n,q)$ are the transfer terms of energy and helicity, respectively. The decay rates, under viscous effects only, of modal energy and helicity, $-2\nu\lambda_{nq}^2H_1(n,q)$ and $-2\nu\lambda_{nq}^2E(n,q)$, are therefore identical. This decay is faster for larger $|\lambda_{nq}|$, i.e., at larger wavenumbers. Consequently, a possible cause of the faster decay of helicity would be that its direct transfer rate is larger than its inverse one, thereafter more helicity would be dissipated (at large wavenumbers). The energy could have the opposite behavior, with a higher inverse transfer rate and thereby a slower dissipation, at small wavenumbers.

It was also shown in Chapter 4 that angular momentum and circulation can be spontaneously generated in our setup, a feature that might be explained by the existence of inverse cascades of these inviscid invariants.

We will test these hypotheses in decaying and in forced turbulence (Sections 5.1 and 5.2, respectively). In the latter case use will be made of a negative viscosity forcing scheme [17].

### 5.1 Decaying turbulence

A spectral analysis of the flow is first performed in the decaying case. This will guarantee that the results obtained are not spurious effects associated with the forcing scheme.

#### 5.1.1 Energy spectrum

We have already recalled that energy displays an inverse cascade in 2D turbulence, and a direct one in 3D turbulence. It is therefore interesting to investigate how it cascades in axisymmetric turbulence, a case intermediate between 2D and 3D.

We will first try to evidence possible direct cascades. For this, initial conditions in which the modal energy is concentrated at the smallest wave numbers must be used. We have considered two typical cases (with or without angular momentum at $t = 0$), as illustrated in the left column of Fig. 5.1. In each case the initial Reynolds number is around 643. The
Fig. 5.1 Modal representation of the initial condition (left), and energy spectra at different times (right), with two different initial conditions at small wavenumbers. Top: zero initial angular momentum; bottom: finite initial angular momentum. Direct cascades are evidenced in both cases.
energy spectrum at different times is shown for these cases in the right column of Fig. 5.1: direct cascades, from the small to the large wavenumbers, are clearly visible. Inertial ranges, that would be characterized by the existence of power law behaviors, cannot be evidenced due to the low values of the Reynolds numbers considered. This will be achieved in forced simulations (see Sec. 5.2).

In order to see if there also exists inverse cascades, we conducted other calculations in which the initial modal energies are concentrated at intermediate values of the wavenumbers (left column of Fig. 5.2): in the first case, direct cascades are permitted both for the axial and for the radial wavenumbers, whereas it is possible only for one or the other wavenumber in the two other cases. The spectra displayed in the right column of Fig. 5.2 show that inverse cascades, from the small to the large wavenumbers, seem to occur in every case. This is also illustrated, mode by mode, in Fig. 5.3.

We have checked that the results previously mentioned are similar in helical and in nonhelical flows. It must be stressed here that the direct and inverse cascades evidenced here are not necessarily energy cascades. To assess this, a detailed flux balance is needed, and we will come back to this in section 5.2.

5.1.2 Helicity spectrum

In 3D isotropic turbulence, specific attention has been paid to the helicity cascade and to its link with the energy cascade [42–44, 64, 89]. We investigate in this section helicity spectra in decaying axisymmetric turbulence.

We will first consider helical flows, in which the total helicity $H_1^{tot}$ has a well defined sign and helicity spectra can therefore be calculated unambiguously. The total helicity spectrum is defined as the net helicity contained within a wavenumber $k = \sqrt{k_n^2 + \gamma_q^2}$:

$$H_1(k) = \sum_{n=0,q}^{\pm \lambda} \sum_{n,q} \xi_{nq} \xi_{nq} \lambda_{nq} + \sum_{n>0,q}^{\pm \lambda} 2 \xi_{nq} \xi_{nq} \lambda_{nq} \quad (\sqrt{k_n^2 + \gamma_q^2} = k).$$

(5.3)

The notation $\sum^{\pm \lambda}$ means that the summation is performed on the modes of positive and negative $\lambda$. We will proceed in the same way as we did for the investigation of energy spectra: two typical initial conditions, in which energy is concentrated at small or at intermediate wavenumbers respectively, will be considered.

The helicity spectrum is shown at different times, for initial conditions illustrated in Fig. 5.1(c) and 5.2(a), in Fig. 5.4. Direct and inverse cascades also appear clearly in these spectra. Inertial ranges cannot be evidenced due to the low value of the Reynolds number considered.
Fig. 5.2 Modal representation of the initial condition (left), and energy spectra at different times (right), with three different initial conditions at intermediate wavenumbers. Top: inverse cascade possible in $q$ (radial wave number) and $n$ (axial wave number); middle: inverse cascade possible in $q$ only; bottom: inverse cascade possible in $n$ only. Inverse cascades are evidenced in every case.
Fig. 5.3 Modal energy transfer at different times, for the initial condition shown in Fig. 5.2(a).
5.1 Decaying turbulence

In nonhelical flows, helicity spectra cannot be plotted straightforwardly since $H_{1}^{\text{tot}}$ does not have a well defined sign, but this quantity can be written as

$$H_{1}^{\text{tot}} = \pm \lambda \sum_{n=0,q} \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq} + \pm \lambda \sum_{n>0,q} 2 \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq}.$$  \hfill (5.4)

This decomposition is very similar to the helical decomposition introduced in the study of isotropic 3D turbulence [42–44, 64, 89, 90]. For nonhelical flows we will investigate the time evolution of the spectra of helicity associated to the two polarities:

$$H_{1}^{+\text{tot}} = \pm \lambda \sum_{n=0,q} \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq} + \pm \lambda \sum_{n>0,q} 2 \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq},$$  \hfill (5.5)

$$H_{1}^{-\text{tot}} = -\lambda \sum_{n=0,q} \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq} + -\lambda \sum_{n>0,q} 2 \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq}.$$  \hfill (5.6)

The notation $\pm \lambda$ (resp. $-\lambda$) means that the summation is performed on the modes of positive (resp. negative) $\lambda$. The spectra $H_{1}^{\pm}(k)$ are defined as the polarized helicity contained within a wavenumber $k$:

$$H_{1}^{\pm}(k) = \pm \lambda \sum_{n=0,q} \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq} + \pm \lambda \sum_{n>0,q} 2 \varepsilon_{nq}^* \varepsilon_{nq} \lambda_{nq} \sqrt{k_n^2 + q_n^2 = k},$$  \hfill (5.7)

Fig. 5.4 Total helicity spectra at different times, in the helical case. (a) Initial condition illustrated in Fig. 5.1(c); (b) initial condition illustrated in Fig. 5.2(a). Direct and inverse cascades are evidenced.

Similar results, not shown here, were obtained with the initial conditions represented in Fig. 5.1(a), 5.2(c) and 5.2(e).
Spectral analysis of the flow

Fig. 5.5 Polarized helicity spectra $H_1^+ (k)$ and $H_1^- (k)$ at different times, for nonhelical flows, with the initial conditions shown in Fig. 5.1(a) (top) and 5.2(a) (bottom).

$$H_1^- (k) = - \sum_{n=0,q} \xi_{nq}^v \varphi_{nq} ^r \lambda_{nq} - \sum_{n>0,q} 2 \xi_{nq}^v \varphi_{nq} \lambda_{nq} (\sqrt{k_n^2 + \gamma_q^2} = k). \tag{5.8}$$

The spectra of polarized helicity have been investigated for several initial conditions (small or intermediate wavenumbers, vanishing or zero angular momentum). They are shown in Fig. 5.5 for two representative cases. A tendency to populate modes associated both to smaller and to larger wavenumbers is clear. However, one cannot straightforwardly interpret this behavior as the signature of cascades, since $H_1^{+, \text{tot}}$ and $H_1^{-, \text{tot}}$ are not separately conserved by the inviscid dynamics (only their sum, the total helicity, is).

### 5.1.3 Spectral analysis of angular momentum

The histories of quantities of our computations in Chapter 4 have shown that the quantity that decays the slowest is in every case the angular momentum $I_1^{\text{tot}}$. The same result holds in 2D turbulence in the presence of circular no-slip rigid walls, and can be then explained as
5.1 Decaying turbulence


the result of an inverse cascade of $I_1^{\text{tot}}$ [31]. We will show here that this is also the case in axisymmetric turbulence.

As shown in Chapter 3, the modes that contribute to the angular momentum are the modes $n = 0$. According to equation (3.49), the modal angular momentum is deduced as

$$I_1(0, q) = (\xi_{0q}^{v, +\lambda} - \xi_{0q}^{v, -\lambda}) \lambda_{0q} \sqrt{2J_2(\gamma_{0q}a)} \gamma_{0q} |J_0(\gamma_{0q}a)|,$$

(5.9)

in which $\xi_{nq}^{v, \pm\lambda}$ means the modal coefficients of eigenfunctions with $\pm\lambda_{nq}$ respectively, as already explained in detail in Chapter 3. We have visualized in Fig. 5.6 the evolutions of modal angular momenta and energies, for the different $(0, q)$ modes, for a typical computation with finite initial angular momentum. According to Fig. 5.6(b), the angular momentum is initially redistributed amongst the different modes. Subsequently the mode $(0, 1)$ becomes dominant (even in cases in which it is initially absent). The same behavior was observed in all the computations carried out with finite initial $I_1^{\text{tot}}$. From the Navier-Stokes equations in spectral space, we can deduce the modal equation of $I_1(0, q)$,

$$\frac{dI_1(0, q)}{dt} = T_{I_1}(0, q) - \nu \lambda_{0q}^2 I_1(0, q),$$

(5.10)

where transfer term is $T_{I_1}(0, q)$ and $\nu \lambda_{0q}^2 I_1(0, q)$ is the dissipation term. The decay rate of the modal angular momentum is proportional to the parameter $\lambda_{0q}^2 = \gamma_q^2$. And we see that the minimum value of $\lambda_{0q}^2$ is from the mode $(0, 1)$ (this is confirmed in Fig. 5.6(c)). Therefore, we propose the following interpretation of the robustness of angular momentum in axisymmetric turbulence: the inverse cascade makes the modal angular momentum transform to the smallest wavenumber, for which the dissipation rate is the smallest.

It is interesting to see that the same cascade and dissipation process will sometimes lead to the “generation” of angular momentum, in the case where the initial angular momentum is zero. We show here that this “generation” is more alike to an accumulation under the coupled effect of cascade and dissipation. As presented in Fig. 5.7, the nonlinear term of Eq. (5.10) initially transfers modal angular momentum into the modes $(0, q)$ (Fig. 5.7(b)): during this period of time ($t/\tau \in [0, 20]$), the total angular momentum oscillates around zero (Fig. 5.7(a)). The decay rates of the modes $(0, q)$ are then different, thus the modes with the larger $q$ will dissipate faster. This may result in an accumulation of angular momentum in the modes like $(0, 2)$ or $(0, 1)$. In other words, during the redistribution of the modal angular momentum, the other larger modes which should balance the smaller modes to maintain net zero angular momentum will dissipate faster. Therefore we can observe a generation of a little amount of angular momentum.
Spectral analysis of the flow

Fig. 5.6 Typical time evolution of angular momentum for \( I_{1}^{\text{tot}}(t = 0) \neq 0\). (a) Total angular momentum; (b) modal angular momentum; (c) modal energy.
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Fig. 5.7 Typical time evolution of angular momentum for $I_1(t) = 0$. (a) Total angular momentum; (b) modal angular momentum; (c) modal energy.
5.1.4 Spectral analysis of circulation

As shown in Chapter 3, the modes that contribute to the circulation are the modes $n = 0$. According to equation (3.51), the modal circulation is deduced as

$$H_0(0, q) = (\xi_{0q}^v + \xi_{0q}^{-} - \lambda) \sqrt{2} \left[ J_0(0) - J_0(\gamma_0 a) \right].$$ (5.11)

In spectral space, the time evolution equation of $H_0(0, q)$ can be formally written as

$$\frac{dH_0(0, q)}{dt} = T_{H_0}(0, q) - \nu \lambda_0^2 H_0(0, q),$$ (5.12)

where $T_{H_0}(0, q)$ is the nonlinear term and $\nu \lambda_0^2 H_0(0, q)$ is the dissipation term. The time evolution of the total circulation and of the modal circulations $H_0(0, q)$ with a finite initial circulation is shown in Fig. 5.8. It is interesting to see that the behavior of the circulation is analogous to that of the angular momentum. The modal circulation will cascade to the small wavenumbers under the effect of the nonlinear term of Eq. (5.12), and the mode $(0, 1)$ will eventually become more dominant because of its slower decay.

Another example is from the computations with zero initial value of $H_0^{tot}$. In Figure 5.9, there is still the inverse cascade phenomenon, the modal circulation will concentrate into smaller wavenumbers (in the absence of viscosity, their sum, the total circulation, would be
5.2 Forced turbulence

Fig. 5.9 Typical time evolution of circulation for $H_0^{tot}(t=0) = 0$. Left: total circulation; right: modal circulation.

conserved). An inverse cascade seems to occur. The mode $(0, 1)$ has the slowest dissipation rate, and circulation seems to be “generated”.

Inverse cascades have been therefore evidenced by the spectral analysis of angular momentum and circulation.

5.1.5 Conclusion

In this section we have performed a spectral analysis of the dynamics of the first inviscid invariants of axisymmetric flows, in decaying turbulence. Direct and inverse cascades have been evidenced in the spectra of energy and helicity. The existence of an inverse cascade explains the robustness and possible generation of angular momentum and circulation. All the results shown are very general and do not depend on the nature of the flow (helical/nonhelical, zero or finite initial angular momentum, ...). We will now investigate these cascades in forced turbulence.

5.2 Forced turbulence

We investigate in this section forced axisymmetric turbulence. This will allow us to consider higher values of the Reynolds number, and in particular to investigate some clear scalings and to measure the transfer terms appearing in the spectra equations of the energy and helicity dynamics.
5.2.1 Description of the forced simulations

5.2.1.1. Methodology

**Negative viscosity method**  A statistically stationary regime was obtained by adding a viscous term (of negative viscosity) in a finite wavenumber band. Considering the forced Navier-Stokes equations,

\[
\partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u + f,
\]

(5.13)

where \( f \) is the external forcing term. The discretization of the Navier-Stokes equations in a spectral space (CK modes) allows to implement a forcing term that acts only in a finite wavenumber band:

\[
\frac{\partial}{\partial t} \xi_{nq}^\nu = \sum_{n'q'} \sum_{n''q''} \lambda_{nq} \lambda_{n'q'} \lambda_{n''q''} \cdot \frac{1}{\nu} \int r dr \int \theta d\theta (\mathbf{A}_{n'q'}^* \times \mathbf{A}_{n''q''}^n) r dr dz \int_\theta d\theta - \nu \lambda_{nq}^2 \xi_{nq}^\nu + \alpha(n,q) \nu \lambda_{nq}^2 \xi_{nq}^\nu,
\]

(5.14)

The last term is the discretized forcing term: \( \alpha(n,q) \) is the forcing amplitude. It is nonzero only in a narrow band of large or intermediate wavenumbers (its value is generally larger than 1).

**Helical and nonhelical flows**  We will hereafter consider two typical configurations of statistically stationary flows: helical flows, with a finite (and therefore of well defined sign) helicity, and nonhelical flows (zero average helicity). One or another regime can be imposed by tuning the forcing term. For this, we will first write the discretized form of the Navier-Stokes equations for the coefficients \( \xi_{nq}^{\nu,+} \) and \( \xi_{nq}^{\nu,-} \) separately: \( \xi_{nq}^{\nu,+} \) and \( \xi_{nq}^{\nu,-} \) are the coefficients associated with the basis of positive and negative \( \lambda \) respectively. A more detailed explanation can be found in Chapter 3. The forcing term contribution in the equations of energy and of helicity can be deduced from equation (5.14):

\[
F_{E_{\text{tot}}} = \sum_{n=0,q} \lambda_{0q}^2 \left( \alpha^{\nu,+} \xi_{0q}^{\nu,+} + \alpha^{-\nu,-} \xi_{0q}^{\nu,-} \right)
\]

\[
+ \sum_{n>0,q} 2 \lambda_{nq}^2 \left( \alpha^{\nu,+} \xi_{nq}^{\nu,+} + \alpha^{-\nu,-} \xi_{nq}^{\nu,-} \right),
\]

(5.15)
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\[ F_{H_1}^{\text{tot}} = \sum_{n=0,q} 2|\lambda_0 q|^3 (\alpha^+ \xi_{nq} + \lambda^+ \xi_{nq} - \alpha^- \xi_{nq} - \lambda^+ \xi_{nq}) + \sum_{n>0,q} 4|\lambda_{nq}|^3 (\alpha^+ \xi_{nq} + \lambda^+ \xi_{nq} - \alpha^- \xi_{nq} - \lambda^+ \xi_{nq}), \quad (5.16) \]

The forcing amplitudes associated to the modes of positive and negative \( \lambda \) are specifically denoted as \( \alpha^+ \lambda \) and \( \alpha^- \lambda \).

On the one hand, in order to impose both stationary and finite energy and helicity, we have set the forcing amplitudes as:

\[ \alpha^+ \lambda = \alpha^- \lambda, \quad (5.17) \]

which is a constant for every coefficients. In this way helicity is forced in the same way as energy.

On the other hand, for the case with finite stationary energy and zero helicity, the forcing term \( F_{H_1} \) needs to be zero. To achieve this, the forcing amplitudes can be determined as:

\[ \alpha^+ \lambda = \alpha^- \lambda \frac{\xi_{nq} + \lambda^+ \xi_{nq}}{\xi_{nq} + \lambda^+ \xi_{nq}}. \quad (5.18) \]

Using this procedure, we managed to force energy but not helicity.

**Friction term**  The existence of an inverse cascade was evidenced in the previous section. In 2D turbulence, in which energy is known to accumulate at the smallest wavenumbers under the effect of forcing, a Reynolds friction must be introduced. It consists in a term \(-\mu u\) added in the Navier-Stokes equations, that allows to dissipate energy mainly in the small wavenumbers. In the cases shown in this thesis, the Reynolds frictions are found to be unnecessary to achieve stationary states. This may imply that the condensation in forced axisymmetric turbulence is not a very serious issue. However, such a friction term would be necessary in the case where the forcing term is implemented at very small wavenumber. This case is not yet studied in our thesis, but it would be interesting to consider it in a future work.

**Numerical setup**  We will consider in this section statistically steady flows at \( Re \sim 888 \), within a cylindrical domain with a rectangular cross section where \( R = \pi \) and \( L = 1.7 \pi \). We have shown that modifying the aspect ratio of the cross section does not affect significantly the results of our computations.
Spectral analysis of the flow

In order to investigate the dual cascade, we have applied the forcing term in a band of intermediate wavenumbers: \( k_f \sim 12 - 15 \). The resulting calculations have successfully reached statistically stationary states. This has been checked by plotting the time evolution of energy. We have also checked that the resolution of our calculations was sufficient by comparing the dissipative scales of different quantities with the maximum wavenumber of the resolution \( k_{\text{eff}} = \max k^2 + \max n^2 q_{\text{max}} \), in the same way as we did for decaying flows.

5.2.1.2. Illustration of the regimes obtained

As already mentioned, we have achieved two typical scenarios. One has a finite stationary total helicity, as shown in Fig. 5.10, the other is a contrast case with a stationary zero helicity as shown in Fig. 5.11. We note the first one as a helical case, and the later one as non-helical case. For the helical case, the time evolution of energy is shown in Fig. 5.10(a). We can see that the system successfully reached the stationary state during \( t \in [5, 30] \). And in Fig. 5.10(b), the total helicity appears to reach a positive stationary value at the same time. In
Fig. 5.11 Illustration of a typical stationary nonhelical flow. (a) Time evolution of energy; (b) time evolution of helicity; (c) comparison of the dissipative scales with the maximum effective wavenumber.

the other scenario, the energy signal is shown in Fig. 5.11(a), and the total helicity is shown to be zero in Fig. 5.11(b). Furtherly, from Fig. 5.10(c) and Fig. 5.11(c) which compare the dissipative scales with the maximum effective wavenumber, we see that during the stationary state, our computation is well resolved.

We now show typical instantaneous flow fields in the stationary states. For the helical case, one example of a stationary field is illustrated in the first row of Fig. 5.12, and for the non-helical case a typical field is illustrated in the second row of the same figure. We see that there is no significant qualitative difference between these two fields. The stream function mainly depends on the largest scales, so that we can observe the growing large-scale structures emerge in the stream function field. From the azimuthal vorticity, we see that structures in different scales are mixing together. We also note that the characteristic scales of the structures are consistent with the fact that most of the energy is concentrated in the wavenumbers $k \in [3, 10]$, a property that will be visible when we will show the corresponding energy spectra. This situation is different from the decaying case, in which the energy
eventually concentrates in the smallest wavenumbers and the final flow rather exhibits large scale structures.

The helical case is similar to the von Kármán flow \[38\], decimated Navier-Stokes turbulence \[44, 91\] and rotating helical turbulence \[92, 93\], in which one sign of helicity is much stronger than the opposite. Meanwhile the non-helical case seems to be closer to isotropic turbulence where no dominant direction of helicity emerges \[43\]. It will be interesting to investigate the spectral statistics of these flows, and that is done now.

5.2.2 Energy and helicity spectra

Energy spectra

The energy spectra associated with the helical and non-helical case are shown at different times in Fig. 5.13. The initial conditions are such that a small amount of energy is given at the forcing scale. The first thing to notice is that the helicity seems to have no obvious effect on the cascade of energy. For both the helical case, and the non-helical case, illustrated in Figure. 5.13(a) and Figure. 5.13(b) respectively, the energy spectra show a dual cascade during the field evolution, and the spectrum tends converge to a steady shape. As for the converged scaling, we see the development of an inverse cascade with a scaling that is roughly compatible to \(E(k) \sim k^{-5/3}\), and a direct cascade with a surprising steep exponent compatible with \(E(k) \sim k^{-5.5}\) (helical case) or even \(E(k) \sim k^{-6.5}\) (nonhelical case). Note that there is no evidence yet to say that the steeper \(-6.5\) slope is only due to the non-helical scenario. The exact cause of the different steep exponent would need to be further investigated, but we will show later that these exponents will all be ‘corrected’ to similar smaller values if one filters the most intermittent regions of the flow. In all, this observation confirms our results in the decaying turbulence where we had seen the tendency of a dual cascade.

It is interesting to see that the exponent of the inverse cascade is close to the prediction of the inverse cascade proposed for the von Kármán experiment \[38\]. And we also recall that this inverse scaling was also obtained in a decimated homogeneous isotropic Navier-Stokes turbulence \[44\] and anisotropic highly rotating helical turbulence \[92, 93\] that have the tendency to generate quasi-two dimensional fields. These observations are compatible with the fact that the inverse cascade in axisymmetric turbulence might be a cascade of energy. Helicity would be more dominant at the largest wavenumbers and seems to exhibit a direct cascade that can block the direct energy cascade and thereby drive energy to larger scales. Such a scenario will be tested in the following subsections, in which we will measure the transfer rates of \(E(k)\) and of \(H_1(k)\). It would be relevant in the helical case but one can wonder how it can explain the scalings observed in the nonhelical case.
5.2 Forced turbulence

Fig. 5.12 Typical instantaneous flow fields in forced turbulence. Top: helical flow; bottom: nonhelical flow.
Fig. 5.13 Energy spectrum at different times: (a) helical case; (b) nonhelical case.

On a side note, it is worth noticing that the helicity and the energy spectra satisfy, by definition, the relation:

\[ H_1(k) \leq kE(k), \tag{5.19} \]

which is formally analogical to the relation \( \Omega(k) = k^2 E(k) \). When helicity is maximal (e.g., in Beltrami flows), the following relation holds:

\[ H_1(k) = kE(k). \tag{5.20} \]

Such a relation indicates that the helicity is more dominant than the energy at large wavenumbers.

As for the direct cascade, we have obtained a slope which seems very steep for both helical and non-helical cases. At these scales, the von Kármán experiment shows a scaling \( E(k) \sim k^{-2} \sim k^{-7/3} \) which is also obtained by a dimensional analysis assuming direct cascades of helicity with non-local and local assumptions respectively. Moreover, the decimated Navier-Stokes turbulence exhibits a (local) direct helicity cascade spectrum compatible with \( E(k) \sim k^{-7/3} \) [44]. And in highly rotational flow, as a result of turning into a quasi-2D turbulence, the energy scaling have also been found to deviate from Kolmogorov scalings. Comparing these results to our observations, \( E(k) \sim k^{-5.5} \) or \( E(k) \sim k^{-6.5} \), our spectra are much steeper than the observation in experiment or any other classical predictions in 2D and 3D. This observation reminds us the early numerical results obtained in 2D turbulence, in which some intermittency 'correction' was proposed to be responsible for the very steep spectra [77]. In the next subsections we will discuss the fact that this steep slope can be 'corrected' by filtering the highly intermittency field areas.
5.2 Forced turbulence

Helicity spectra

As already noticed, the spectrum of helicity cannot be trivially defined if $H_1^{\text{tot}}$ does not have a well defined sign (non-helical case). In our numerical method, the total helicity can be straightforwardly written as the sum of components respectively representing the positive and negative polarities. Firstly, according to the equation (3.48) in Chapter 3, the energy associated to the two polarities can be shown as

$$E^{\pm, \text{tot}} = \sum_{n=0,q}^{+\lambda} \frac{1}{2} \xi_{nq}^v \xi_{nq}^v + \sum_{n>0,q}^{+\lambda} \xi_{nq}^v \xi_{nq}^v,$$

(5.21)

$$E^{-, \text{tot}} = -\sum_{n=0,q}^{-\lambda} \frac{1}{2} \xi_{nq}^v \xi_{nq}^v - \sum_{n>0,q}^{-\lambda} \xi_{nq}^v \xi_{nq}^v.$$

(5.22)

The spectra $E^{\pm}(k)$ are defined as the polarized helicity contained within a wavenumber $k$:

$$E^+(k) = \sum_{n=0,q}^{+\lambda} \frac{1}{2} \xi_{nq}^v \xi_{nq}^v + \sum_{n>0,q}^{+\lambda} \xi_{nq}^v \xi_{nq}^v \left( \sqrt{k_n^2 + \gamma_q^2} = k \right),$$

(5.23)

$$E^-(k) = -\sum_{n=0,q}^{-\lambda} \frac{1}{2} \xi_{nq}^v \xi_{nq}^v - \sum_{n>0,q}^{-\lambda} \xi_{nq}^v \xi_{nq}^v \left( \sqrt{k_n^2 + \gamma_q^2} = k \right).$$

(5.24)

And the relation between the total energy spectrum $E(k)$ and the ones with polarities is

$$E(k) = E^+(k) + E^-(k).$$

(5.25)

Now the spectra in polarities of helicity have already been shown in equations (5.7) and (5.8), hence we have the relation:

$$H_1(k) = H_1^+(k) - H_1^-(k) = 2\lambda_{nq} E^+(k) - 2\lambda_{nq} E^-(k),$$

(5.26)

where we have noted $\lambda_{nq} = \sqrt{k_n^2 + \gamma_q^2}$ (the effective wavenumber is $k = |\lambda_{nq}|$). The relation above shows that $H_1^+(k)$ (resp. $H_1^-(k)$) represents the helicity contribution of the modes of positive (resp. negative) helicity. Here we have the relation between modal helicity and modal energy that

$$H_1^\pm(k) = \pm 2k E^\pm(k).$$

(5.27)

Hence the modal quantities with polarities is connected to the $E(k)$ and $H_1(k)$ as

$$E^+(k) + E^-(k) = E(k),$$

(5.28)
Fig. 5.14 Helicity spectra in the helical case. (a) Total helicity; (b) positively polarized component; (c) negatively polarized component.

\[ 2kE^+(k) - 2kE^-(k) = H^+_1(k) - H^-_1(k) = H_1(k). \]  \hspace{1cm} (5.29)

We need to note here that these expansions of two parities are very similar to the decomposition of helical waves [42, 43, 64, 89] for incompressible velocity field, and here the conventions are also very similar to the work [43].

We first show in Fig. 5.14(a) the total spectrum of helicity (well defined) in the helical case, at different times. Direct and inverse cascades are clearly visible. The exponents of these cascades are found to be compatible with \( H_1(k) \sim kE^\pm(k) \). In the nonhelical case we have plotted in Fig. 5.15 the spectra \( H^+_1(k) \) and \( H^-_1(k) \). As a comparison, the same quantities are plotted in Fig. 5.14(b) and (c) for the helical flow. In every case the relation \( H^\pm_1(k) \sim kE^\pm(k) \) seems to hold.
5.2 Forced turbulence

Finally, we have plotted in Fig. 5.16 the spectra of energy for the two polarities, in the non-helical case. A dual cascade with the same exponents as in Fig. 5.13(b) is observed, indicating that approximatively $E^+(k) \sim E^-(k) \sim E(k)$ for both the inverse and the direct cascade. Such an observation trivially satisfies equation (5.28). Since we can deduce the relation between the polar spectra and the total ones as

$$E^\pm(k) = \frac{1}{2} \left( 2E(k) \pm \frac{H_1(k)}{k} \right),$$

(5.30)

this implies that the spectrum of helicity $H_1(k)$ also scales as $kE(k)$. Similar results (not shown here) were obtained in the helical case, that is $E^+(k) \sim E^-(k) \sim E(k)$ for the two cascades.

---

Fig. 5.15 Helicity spectra in the non-helical case. (a) Positively polarized component; (b) negatively polarized component.

Fig. 5.16 Energy spectra in two polarities in the non-helical case. (a) Positively polarized component; (b) negatively polarized component.
Fig. 5.17 Time evolution of the energy components in the stationary state. (a) Helical flow; (b) non-helical flow.

### 5.2.3 Spectra of the toroidal and poloidal components of the flow

We hypothesized in the previous decaying fields that the overall dynamics of axisymmetric turbulence might be explained by analyzing separately the different cascade behaviors of poloidal and toroidal components. In this subsection we investigate this further. We recall that these two components obey the following equations, derived by [20, 24], in the absence of forcing and of viscosity:

\[
\frac{\partial \sigma}{\partial t} + \{ \psi, \sigma \} = 0, \quad (5.31)
\]

\[
\frac{\partial \xi}{\partial t} + \{ \psi, \xi \} = \frac{\partial}{\partial z} \left( \frac{\sigma^2}{4y^2} \right). \quad (5.32)
\]

These two equations represent the control equations of the toroidal related component \( \sigma \) and the poloidal related component \( \xi \). It can be first noticed that equation (5.31) is an advection equation, similar to that describing the scalar transport of a quantity \( \sigma \) in the 2D plane \((r, z)\). This implies that the toroidal component is mainly expected to cascade to the small scales. Meanwhile, the poloidal equation (5.32) possesses an extra source term \( \frac{\partial}{\partial z} \left( \frac{\sigma^2}{4y^2} \right) \) from the toroidal component. Despite of this source term the poloidal equation is similar to the 2D vorticity equation, which may imply the existence of a dual cascade, in the two directions. Strictly speaking, considering cascades of poloidal and toroidal energy is abusive since these quantities are not inviscid invariants separately (only their sum is). Our arguments here are thus purely phenomenological.

We first calculated the different energy components \( u_r^2/2, u_\theta^2/2 \) and \( u_z^2/2 \) during the stationary stage of our simulations. The time signals of these quantities is shown in Fig. 5.17(a) in the helical case. Under such a forcing we notice that the poloidal energy is more
Fig. 5.18 Toroidal and poloidal energy spectra in the helical case.

dominant than the toroidal energy. The two toroidal components, $u_r^2$ and $u_z^2$, behave similarly. This figure indicates that the forcing method we applied needs to be improved if one wants to inject more energy in the toroidal direction. Whatsoever, the amount of toroidal energy is sufficient to observe a visible spectrum. In Fig. 5.17(b) the same quantities are plotted in the non-helical case. We find that the forcing method which aims at giving no input to the net helicity also gives no input to the toroidal energy. Therefore in this later case we are already in a stationary quasi-2D turbulence, which implies a tight connection between the net helicity and the toroidal energy.

In order to see the transport behaviors of these two components, we have plotted in Fig. 5.18 the toroidal ($E_t(k) = u_\theta^2(k)/2$) and poloidal ($E_p(k) = (u_r^2 + u_z^2)(k)/2$) energy spectra in the helical case (in the nonhelical case there is no toroidal energy). As already shown, the toroidal energy is much lower than the poloidal one. The direct cascade is visible in both spectra, with a scaling compatible with that exhibited in the total energy spectrum. It is noticeable that the inverse cascade is visible in the spectra $E_p(k)$ but not as clearly in the spectra $E_t(k)$. So far this appears to show that the toroidal component indeed mainly cascade to small scales as we presumed in axisymmetric turbulence, meanwhile the poloidal components are more like 2D with a dual cascade. The results here give some hints of a possible interesting property of axisymmetric turbulence: the inverse cascade phenomenon would be mostly associated with the quasi-2D $r-z$ plane. This would provide an evidence that the statistical approach of 2D is essential in the approach of axisymmetric turbulence.

### 5.2.4 Energy and helicity transfer rate

In the helical case, there is both net nonzero stationary energy and helicity. Hence it would be interesting to investigate the transfer rate of these two invariants. We have already mentioned
the fact that a possible scenario would be a dual cascade of energy (inverse cascade, at large scales) and of helicity (direct cascade, at small scales). We now measure the transfer rates of these two quantities to show that this scenario is plausible.

From the discretized form of the Navier-Stokes equations, we can deduce the standard spectral balance equation for energy and helicity,

\[ \partial_t E(n,q,t) = T_E(n,q,t) - 2\nu \lambda_n^2 E(n,q,t) + F_E(n,q,t), \]

and

\[ \partial_t H_1(n,q,t) = T_{H_1}(n,q,t) - 2\nu \lambda_n^2 H_1(n,q,t) + F_{H_1}(n,q,t), \]

where \( T_E \) and \( T_{H_1} \) are the transfer functions, \( F_E \) and \( F_{H_1} \) are the forcing contributions. Since energy and helicity are both invariants, the transfer functions satisfy the integral relations:

\[ \int_0^\infty dk T_X(k,t) = 0, \]

where \( X = E, H \) respectively state the conservations of energy and helicity. One also has

\[ \Pi_X(k,t) = -\int_0^k T_X(k',t) dk', \]

where \( \Pi_X \) denotes the transfer rate or flux of quantity \( X \) through wave number \( k \). Thereafter, in order to see more clearly the energy flux of the dual cascade flow, we have calculated the average of this quantity during the time interval of stationary state \( t = [10,30] \). This quantity is plotted in Fig. 5.19(a). We can see first that two directions of transfer are visible: in average there is respectively a negative flux at the wavenumbers smaller than \( k_f \), and a positive one at \( k > k_f \). It appears that our resolution is still too small to observe a constant transfer rate which would indicate a clear inertial range. Nevertheless, by comparing these two flux directions of energy, we can already see that the inverse energy flux is more dominant than its own direct flux.

In Fig. 5.19(b), the helicity flux averaged in the same time interval is shown. Inverse and direct helicity flux are both observed. Once more, the limited Reynolds number of our calculation does not allow to observe inertial ranges (constant flux), but we see that the direct transfer rate is more dominant than the inverse one, a behavior opposite to that of energy. We have performed another simulation with finite helicity and obtained the same qualitative results. All in all, Fig. 5.19 supports the idea that, in the helical case, a dual cascade occurs (direct cascade of helicity and inverse cascade of energy). This scenario explains why helicity decays much faster than energy in the unforced simulations.
5.2 Forced turbulence

In the non-helical case, since there is no net helicity, it is interesting to investigate the flux rate of $E^+(k)$, $E^-(k)$, $H_1^+(k)$ and $H_1^-(k)$. Our conventions of polarities are very alike the work of [43], hence referring to their analytical results, we have also deduced the relations of the transfer functions as following,

$$T_{E^±}(k,t) = \frac{1}{2} \left[ T_E(k,t) ± \frac{T_{H_1}(k,t)}{2k} \right],$$  \hspace{1cm} (5.37)

$$T_{H_1^±}(k,t) = \frac{1}{2} \left[ 2kT_E(k,t) ± T_{H_1}(k,t) \right],$$  \hspace{1cm} (5.38)

where $T_{E^±}(k,t)$ is the transfer function of $E^±(k)$, and $T_{H_1^±}(k,t)$ is the transfer function of $H_1^±(k)$. Then the flux rates associated with these transfer functions can be defined as

$$Π_{E^±}(k,t) = - \int_0^k T_{E^±}(k',t)dk',$$  \hspace{1cm} (5.39)

$$Π_{H_1^±}(k,t) = - \int_0^k T_{H_1^±}(k',t)dk',$$  \hspace{1cm} (5.40)

where $Π_{E^±}(k,t)$ and $Π_{H_1^±}(k,t)$ are the flux rates through the wavenumber $k$. Note that because the transfer functions in polarities do not necessarily equal zero with the integration of $k = 0 \sim ∞$, the flux rates here may not be zero when $k = ∞$. We have calculated the average of these flux quantities during the stationary state over the time interval $t = [16, 26]$, so that the flux rates of the quantities associated with the two polarities can be seen. In Figure. 5.20(a) and 5.20(b), we find here also the dual cascade flow for the energy in both polarities. There is a small portion of nearly constant inverse flux observed, yet considering our current
resolution, this can not be concluded to be an inertial range. Whatsoever, the inverse flux is still found to be more important than the direct flux. Moreover, the two fluxes behave indentically, implying the same cascade dynamics for both of the polarities.

From Figure 5.20(c) and 5.20(d), the averaged helicity flux in both polarities are calculed within the same time interval. Once again we observe a more dominant direct flux than the inverse one. Even though there is no net helicity in the field, the helicity in polarities still occupy the small scales simultaneously and drives the energy of the same polarities using an inverse cascade to the large scales. This is an interesting observation that the helicity in polarities can affect the energy cascade separately in axisymmetric turbulence.

We have already shown that scaling laws of the energy spectra were in agreement with this scenario for the inverse cascade ($E(k) \sim k^{-5/3}$), but not for the direct cascade ($E(k) \neq k^{-7/3}$). We will now show that the expected scaling can be recovered by filtering the flow.
Fig. 5.21 Probability distribution functions of the normalized velocity components, measured during the statistically stationary state. (a) Helical case, (b) non-helical case.

5.2.5 Effect of filtering

Statistics of the velocity components in the physical space

The energy spectra display very steep exponents at large wavenumbers \( E(k) \sim k^{-5.5} \) or even \( k^{-6.5} \). Although there is no classical scaling theory for the axisymmetric turbulence, one would expect that the general arguments based on dimensional analysis and on the existence of constant fluxes of energy and of helicity \([38, 44]\), which lead to expectations \( E(k) \sim k^{-7/3} \), allow to interpret our results. The presence of such large exponents reminds us the early observations made in the simulations of 2D turbulence. These were explained by the small resolution of the calculations. In these regimes, the spatial and temporal intermittency areas in the field may play an important role in the shifting of the exponent (a detailed discussion of the model proposed by \([77]\) is provided in Chapter 2), and filtering out the flow allowed to get spectra in better agreement with theoretical expectations \([80]\). We will now try to filter out the flow, to see if the same idea is valid in axisymmetric turbulence.

Before doing so, we measure the probability density functions (PDF) of \((u_r, u_\theta, u_z)\) during \( t \in [10, 30] \). Fig. 5.21 shows these PDFs in the helical and non-helical cases. Each component is normalized with its root mean square. In the non-helical case, only the poloidal components are considered since we have already shown that the toroidal component was negligible. Even though we may not have sufficiently converged statistics, we still see that all the PDFs have much larger tails than the Gaussian distribution, which means that there are noticeable areas of the flow in which relatively large values of the velocity are encountered. According to the explanation of \([77]\), we presume that the domain in the flow field with much larger intermittent velocity value may have a impact on the slope of the energy spectra, and that the latter may therefore be corrected by considering only the rest of the domain,
that is the values of $u_r$, $u_\theta$ and $u_z$ smaller (in absolute value) than their root mean squares: $|u_r| < \sigma_{u_r}$ and $|u_\theta| < \sigma_{u_\theta}$ and $|u_z| < \sigma_{u_z}$.

**Effect of the filtering on the energy and helicity spectra**

The results of corrected energy spectra in the helical case are shown in Fig. 5.22(a), and the corrected helicity spectra for the same flow are illustrated in Fig. 5.22(b). In total, the surprising outcome is the change of slopes of the spectra at small scales. We see that by eliminating the actively intermittent areas, the energy spectra exhibit at large wavenumbers slopes compatible with the dimensional prediction $E(k) \sim k^{-7/3}$. Meanwhile, the relation $H_1(k) = kE(k)$ still holds: at large wavenumbers $H_1(k) \sim k^{-4/3}$.

The “corrected” energy spectra, together with the spectra of positive and negative contributions to the helicity, in the non-helical cases are shown in Fig. 5.23. There are two interesting results here: the first one is the change of slope at large wavenumbers of both the energy and helicity spectra, as we observed in the helical case. The other is that the corrected spectra are now much more similar to those obtained in the helical case (we indeed remember that the uncorrected exponent was slightly steeper in the nonhelical than in the helical case).

These observation reveal the possible correlations between the intermittency and the direct cascade. Even though the model of [77] is not verified directly, we do notice the ‘correction’ of spectrum by eliminating the influence of intermittent areas. The inverse cascade is interestingly found not to be affected by the filtering. The exponent of the energy spectra at small wavenumbers have the value near $-5/3$. Such a scaling is in agreement with the results derived from the inverse energy cascade and the scenario of a dominant helicity direct cascade [38]. The corrected dual cascades appear to be consistent with the observation $H_1(k) \approx kE(k)$. 

---

Fig. 5.22 (a) Energy spectra and (b) helicity spectra in the helical case obtained after a filtering of the flow: $|u_r| < \sigma_{u_r} \cup |u_\theta| < \sigma_{u_\theta} \cup |u_z| < \sigma_{u_z}$. 
Fig. 5.23 Spectra, in the nonhelical case, obtained after a filtering of the flow $|u_r| < \sigma_{u_r}$ $\cup$ $|u_z| < \sigma_{u_z}$: (a) energy spectrum, (a) positive contribution to helicity, (c) negative contribution to helicity
Fig. 5.24 (a) Poloidal and (b) toroidal energy spectra in the helical case, obtained after a filtering of the flow $|u_r| < \sigma_{u_r} \cup |u_\theta| < \sigma_{u_\theta} \cup |u_z| < \sigma_{u_z}$.

**Effect of the filtering on the poloidal and toroidal spectra**

In the helical case, we already seen clearly the different behaviors exhibited by the poloidal and toroidal energy spectra at small wavenumbers. We show here that the slopes of these two spectra are ‘corrected’ in the similar way under the effect of filtering. In Fig. 5.24, it is obvious to see that the exponents at large $k$ are consistent with that of the total energy spectra, that is $E_p(k) \sim E_t(k) \sim k^{-7/3}$, whereas the large scale behavior has, similarly, not been affected by the filtering.

**5.3 Summary**

In this chapter, the spectral analysis of axisymmetric turbulence at moderate Reynolds number has been performed. We first considered decaying fields. The energy and helicity spectra measured in this regime show the existence of a dual cascade. An inverse cascade seems to be responsible of the robustness of angular momentum, and of the possible spontaneous “generation” of angular momentum and of circulation in the flow. The mechanism seems to be identical to that previously identified in 2D turbulence with circular no-slip rigid walls.

We then considered statistically stationary turbulence. Our numerical method allows to consider two distinct regimes: a helical one and a nonhelical one. We showed that stationary states could be obtained in both cases without adding a friction term in the Navier-Stokes equation. The scales of the structures in the flow naturally depend on the wavenumbers at which the most energy is contained. A clear dual cascade was evidenced in both cases. Helicity spectra were plotted in the helical case. In the non-helical one we plotted instead the spectra of the positively and of the negatively polarized helicity components. The
relation $H_1(k) \sim kE(k)$ holds in every case, which means that, at least in the helical flow, helicity should dominate at large wavenumbers, which would be compatible with the possible existence of a direct cascade of helicity and an inverse cascade of energy.

This behavior has been interpreted as a possible consequence of the different cascade behaviors of the poloidal and toroidal components of energy (even though one should be careful with this kind of argument since these two quantities are not separately conserved by the inviscid and unforced dynamics). We found that the toroidal component might mainly exhibit a direct cascade (its formally obey the equation of a passive scalar in the inviscid and unforced case), while the poloidal component has a quasi-2D dynamics and might exhibit a dual cascade (it was incidentally found to be more conservative in the decaying case). From the point of view of a statistical approach, this may reveal that the statistical theory of 2D turbulence may play an important role for understanding axisymmetric turbulence. The flux of helicity and of energy were then shown for the helical case. Despite the low Reynolds number of our simulations, we showed that the helicity is more transported to the small scales than to the large ones, and that meanwhile the energy has the opposite behavior, thereby evidencing direct (respectively inverse) cascades of helicity (respectively energy). This explains why helicity was found to be a very fragile invariant in the decaying simulations.

The scaling of the energy spectra is compatible with the existence of an inverse energy cascade at the smallest wavenumbers, but it is not with that of a direct helicity cascade at the largest wavenumbers. Like in early simulations of 2D turbulence, the steep slopes that we obtained seem to be due to the intermittent areas of the flow. By filtering these areas, we have obtained the energy spectra with a scaling in agreement with the theoretical expectation. The inverse cascade is nearly unaffected by this filtering.

We have considered a case in which the toroidal component of the flow is much smaller than the poloidal ones. It would be interesting to investigate situations in which both components are of the same order, or in which the former is larger than the latter. Due to the moderate Reynolds numbers considered in our simulations, the exact values of the cascade exponents could not be measured accurately and a constant flux in a large band of wavenumbers was not obtained. Higher Reynolds numbers would be needed to address this.
Chapter 6

Conclusion and Perspectives

6.1 Conclusions

The initial goal of this thesis was to explore by numerical simulations the dynamics of axisymmetric turbulence. Being the first simulations of this type of turbulence, and considering the large parameter space, the nature of the present investigation is necessarily exploratory. This exploration allowed a number of interesting robust features of axisymmetric turbulence to be discovered, in particular considering the tendency towards poloidalization of the flow, and the dual cascade of energy and helicity. We will now discuss the specific results and insights presented in this manuscript.

Numerical method. The adaptation of an existing two-dimensional code of the Navier-Stokes equations projected on Chandrasekhar-Kendall eigenfunctions of the curl was successfully carried out, resulting in an original tool to study axisymmetric turbulence. The quite natural decomposition on helical modes allows to impose incompressibility exactly, conserve energy and helicity down to machine precision, and to compute nonlinearities without aliasing effects. The spectral nature of the method allows the implementation of a spectrally localized forcing, and allows to evaluate the contribution of different modes on the different quantities without ambiguities. It further simplifies the generation of initial conditions with a given amount of helicity and/or angular momentum. The natural drawback of the method is the rapidly increasing computational cost (with increasing resolution) associated with the convolution products representing the nonlinearity of the equation. A further drawback is that the axial velocity component does not satisfy the no-slip boundary condition at the wall of the cylinder.
Self-organization of axisymmetric turbulence. In simulations of freely decaying axisymmetric turbulence, it is shown that angular momentum is the most robust invariant. Helicity seems to decay faster than angular momentum and energy. Large-scale coherent structures are observed, mainly associated with the poloidal components of the velocity. This poloidal velocity advects the toroidal component, which behaves as a passive scalar, once it becomes weak enough.

The poloidalization of the flow leads to a two-dimensional dynamics, and the long-time coherent structures are compatible with predictions from Miller-Robert-Sommeria statistical mechanics. However, these structures are different from the Beltrami-structures predicted for axisymmetric flows. It is possible that this disagreement disappears for higher Reynolds numbers and different initial conditions. But for the moment, the understanding of the system is that helicity is a weak invariant, less robust than the energy, unlike the assumptions in theoretical work on axisymmetric turbulence.

Cascades and fluxes. Our numerical method allows a precise modal decomposition of the flow and the global quantities. We have therefore attempted to better understand why certain quantities are better conserved than others, first in freely decaying, later in forced simulations. We first show that, like in two-dimensional turbulence in a circular domain, an inverse cascade is associated with the robustness of the angular momentum. Indeed, angular momentum is transferred to the least dissipative mode, and thereby becomes the most robust of the different invariants in our simulations.

A further study of the modal dynamics showed that the energy tends to cascade to the large scales, whereas the helicity is transferred to the small scales. This illustrates why energy is more robust in the decaying simulations than the helicity. Even in the absence of a net helicity, the same mechanism seems to persist. It is now not the net-helicity which is transferred, but an amount of ’local’ helicity. This was illustrated by considering the decomposition of the flow on our Chandrasekhar-Kendall modes, which are a family of helical modes of two signs. The transfer of an amount of helicity of a certain polarity seems to follow the same mechanism as the transfer of the total helicity in the case of a helical flow. The difference is that helicity associated with a single polarity is not a conserved quantity of the nonlinear interaction. Nevertheless, the helicity cascade seems to be a robust feature, even in the absence of net-helicity.

The powerlaw exponent of the kinetic energy spectrum in the inverse cascade range is compatible with a $-5/3$ value obtained from dimensional analysis, assuming local transfer. In the forward cascade range one would expect by the same arguments a $-7/3$ powerlaw exponent of the wavenumber spectrum. This is not observed and a much steeper spectrum
proportional to $k^{-6}$ is present in this range of wavenumbers. It is however shown that if the strongest intermittent features of the flow field are filtered out, a close to $-7/3$ powerlaw appears for the energy spectrum of the filtered field in the forward cascade range.

### 6.2 Perspectives

The present thesis has opened a whole new playground for turbulence research, investigating a new type of turbulent flows. For instance, considering the amount of studies that have been carried out on two-dimensional turbulence since the work of Onsager in the 1940s, it is probably an understatement to say that there are lot of perspectives for the present study.

**Towards a more efficient numerical method.** It is for the moment not clear whether freely decaying axisymmetric turbulence will self-organize to form the structures which were observed in the Von Kármán experiment. Which values of the integral quantities would be needed for that? And how high must the Reynolds number be?

In any case will it be necessary, if we want to address these questions, to use a different numerical method. The present method, even using increased numerical power, will remain limited in resolution to a Reynolds number of a few thousands. To go beyond this, retaining spectral precision, we will need a pseudo-spectral solver. One possibility is to use a pseudospectral Cartesian method combined with a penalization technique to impose the boundary conditions [94, 95]. Such methods allow a large flexibility in the shape of the domain but loose their spectral precision due to the penalization technique. Since our perspectives consider cylindrical geometry only, flexibility is not so much required. Our direct perspective is to use a pseudospectral code in cylindrical geometry, of the type described in references [96, 97], to allow Reynolds numbers at least an order of magnitude larger than the ones presented in the present work. These codes also allow to use more realistic boundary conditions on the walls of the cylinder, approaching thereby more closely the experimental results.

**Open questions.** The number of open questions is obviously far more important than we will address in this paragraph. We will ask some, in particular about what can be possibly observed at higher Reynolds numbers.

Obtaining flows at higher Reynolds numbers will allow to test the robustness of the results presented in this thesis, assess if a Beltrami flow is observed at higher Reynolds numbers. Indeed, Beltramization was better observed in the experiments at the highest Reynolds numbers. We can further check if the poloidal flow, even initially small, will always
dominate at long times, if the Reynolds number is sufficiently large. Will condensation of energy at the smallest poloidal scale determine the large scale dynamics? Is it useful to add a linear friction to the dynamics?

With respect to the spectral dynamics, higher resolutions might allow to show whether the scaling exponents in the direct cascading range become closer to $-7/3$ even without filtering. Will the fluxes, as a function of wavenumber become constant, for high enough Reynolds numbers.

Another interesting perspective, even at the Reynolds numbers attainable with the present numerical method, is the direct forcing of the toroidal component, with and without injecting helicity. This would allow to check if we can approach the experimental situation some more, where through the impellers, energy is also mainly injected into the toroidal flow component, together with a substantial amount of helicity.

It seems that, to assess the validity of statistical mechanics applied to axisymmetric turbulence, a substantial amount of further research is needed!
Appendix A

Assessment of a minimization principle for three-dimensional turbulence and scalar mixing

A.1 Introduction

As described in chapter 2, statistical mechanics have been applied successfully to two-dimensional turbulence. In three-dimensional turbulence such efforts have been inconclusive for so far. Indeed, the vortex stretching term, which distinguishes the vortex-dynamics in 3D turbulence from the 2D case, allows the destruction or generation of enstrophy in the limit of zero viscosity. Enstrophy is therefore not longer an invariant of the equations. The remaining invariants of 3D turbulence are the kinetic energy and helicity. However, the dissipation of kinetic energy does not tend to zero for vanishing viscosity (an observation also called dissipative anomaly), so that at high Reynolds numbers the kinetic energy is not even approximately conserved. The other invariant, helicity, is shown to cascade towards small scales [43, 61, 64, 89] and does not seem to be conserved either.

It does therefore not seem to be possible to transpose directly the ideas of 2D turbulence to the 3D case, since the inviscid case is of a completely different nature compared to the viscous case. The question that we want to address in this chapter is the following: even in the absence of conserved invariants, is it possible to formulate a optimization principle for three-dimensional turbulence and mixing, based on other global quantities? This can obviously not be verified in the rigorous framework of statistical mechanics, and we will therefore carry out this study using phenomenological selective decay principles.
A.1.1 Selective decay and non-conserved quantities

In chapter 4 we have focused on the possibility that selective decay principles can describe and predict the evolution of axisymmetric turbulence. Some evidence was found for particular cases. Similar variational problems, which in some particular cases are equivalent to the maximization of a mixing entropy under some specific constraints, had been previously proposed to describe the evolution of 2D [24] and axisymmetric Euler flows [25]. In these cases the predictions were obtained by determining extrema of a functional, after the method of Lagrange multipliers was used. The possible validity of selective decay principles, and their relation with maximum entropy states, allows for a phenomenological interpretation of equilibrium statistical mechanics as a special case of the more general, but less rigorous, selective decay principle. Such principles are more easily tractable in practice.

In the formulation of selective decay principles, the whole question is which invariants are more robust. For instance, in the works trying to explain the observations in the von Kármán flow [24], it was conjectured that helicity was the robust invariant. The results in Chapter 5 seem to indicate that the opposite occurs in our simulations of axisymmetric turbulence (carried out at moderate Reynolds numbers). There is no rigorous way to predict which invariants are the most robust ones, but several ways to determine this are proposed. For instance, in the context of two-dimensional turbulence invariants which are not affected by coarse-graining are more robust than the others [9, 48]. Other arguments are based on cascade directions: if an invariant quantity tends to cascade towards dissipative scales, it is likely to be less robust than invariants which cascade away from it. If the dissipative scales are defined as the small scales, then direct cascades are related to more fragile invariants than the inverse cascading invariants.

This shows the main problem in extending the statistical mechanics framework to 3D: in this case there are no robust invariants. Since both inviscid invariants of homogeneous and isotropic 3D flows seem to be fragile, the question is whether we can identify another property of the flow, which is more robust than those invariants. We thereby leave the more rigorous framework of invariants, and we are looking for global flow properties of another nature. The quantity which we focus on in this chapter is related to a feature which has been observed in a certain number of different flows and systems: the tendency of a system to relax to a force-free state (see for instance section 3.1), or at least to a state in which the nonlinearity is reduced.
The suggestion that a turbulent flow tends to a state depleted from nonlinearity was suggested and numerically verified by [98]. Indeed, the authors observed some tendency for Beltramization in turbulent channel flow. This Beltramization was not confirmed in subsequent works [99], but it was observed that the mean-square nonlinearity tends to a value below its Gaussian estimate. This was verified in decaying 3D turbulence and conjectured to be a general feature of nonlinear systems [45]. It was recently confirmed in forced 3D turbulence [100], and decaying 2D turbulence [101]. Also, the von Kármán experiment, where the results of axisymmetric equilibrium states were verified for the mean field, is compatible with a flow where the time-averaged profile is of the Beltrami type, free from nonlinearity [22].

That this observation is compatible with two-dimensional turbulence and long-time equilibrium states, is clear. Indeed, in the two-dimensional Navier-Stokes system, the state corresponding to the Equilibrium-statistical-mechanics state of the Euler equations, is a state in which the nonlinearity vanishes. We could reformulate the problem by trying to minimize the strength of the nonlinear term under some integral constraints. We illustrate that here using the Euler-Lagrange formalism. If we want to find extrema of a quantity $L = \int dx \, dy \, f[u(x,y), v(x,y), x, y]$ depending on the functions $u(x,y)$ and $v(x,y)$, the variational problem can be restated as

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial (\partial u/\partial x)} - \frac{\partial}{\partial y} \frac{\partial f}{\partial (\partial u/\partial y)} = 0$$

$$\frac{\partial f}{\partial v} - \frac{\partial}{\partial x} \frac{\partial f}{\partial (\partial v/\partial x)} - \frac{\partial}{\partial y} \frac{\partial f}{\partial (\partial v/\partial y)} = 0. \quad (A.1)$$

The difficulty is still to find the proper quantity to optimize. Let us consider for two-dimensional turbulence that we look for a force-free state, for a given energy and enstrophy. In this case $L$ will then be given by

$$L = \int dx [\{\psi, \omega\}^2 - \beta \psi \omega - \gamma \omega^2]. \quad (A.2)$$

Minimizing this quantity, for variations in $\omega$, we find after some algebra,

$$\{\psi, \{\psi, \omega\}\} - \beta \psi - \gamma \omega = 0. \quad (A.3)$$

It is clear that the solution $\psi = -(\gamma/\beta) \omega$ is a solution to this equation. The minimization of the square of the nonlinear term is therefore compatible, to first order with observed stationary solutions of the Euler equation. It is not clear that this linear relation is the most
general solution of Eq. \( (A.3) \). However, we think that it is at least worth to try to see, whether a selective decay principle based on the strength of the nonlinearity can be applied to the case of three dimensional turbulence, since the depletion of nonlinearity seems to be a robust feature (observed in both 2D and 3D flows) and since it is compatible with the observed equilibrium solutions of 2D turbulence.

As a first step, we consider here the simpler case of the advection of a passive scalar. Since in this cases the depletion of the advection term behaves rather similar to the depletion of nonlinearity \([46, 102]\), we will focus on this case which is simpler, since the passive scalar is governed by a simple advection-diffusion equation, without the complications of a pressure field to impose incompressibility.

### A.2 Minimization of the advection term in three-dimensional turbulent mixing

#### A.2.1 Euler-Lagrange equations for a minimized advection term

The quantity which we conjecture to be optimized under constraints is the mean-square advection term compared to its Gaussian estimate.

\[
\min \left\{ \frac{\langle (u \cdot \nabla \theta)^2 \rangle}{\langle (u \cdot \nabla \theta)^2 \rangle_{\text{Gaussian}}} \right\}. \tag{A.4}
\]

In a recent investigation, it was shown that the depletion of advection was directly related to the tendency of the scalar gradient to align with the local velocity field \([103]\). In order to measure the rate of alignment quantitatively, we compare to the alignment of a Gaussian scalar gradient field with a Gaussian velocity field, both being described by a spectral distribution identical to that of the turbulent field. A reduction of advection is then associated with the observation that this variance in the turbulent field will tend to be smaller than its Gaussian counterpart. We simplify therefore our problem by focusing on the cosine of the angle between the scalar gradient and the velocity in the following, this angle is given by

\[
\cos \phi = \frac{(u \cdot \nabla \theta)}{((u)^2(\nabla \theta)^2)^{1/2}}. \tag{A.5}
\]

The Lagrangian we consider should represent the minimization of the cosine between \( u \) and \( \nabla \theta \), for given integral constraints. Possible constraints are the scalar variance and scalar dissipation. Obviously this list may not be exhaustive, but these quantities seem to be the
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most obvious choices. Using the Lagrange multipliers method, the minimization problem we consider is therefore,

\[ \min_\theta \{ \langle \cos^2 \phi \rangle | \langle \theta^2 \rangle \langle (\partial_i \theta)^2 \rangle \}. \quad (A.6) \]

The functional \( L \) in this case is

\[
L = \int d\mathbf{x} \{ \cos^2 \phi - \gamma \theta^2 - \eta (\partial_i \theta)^2 \}, \quad (A.7)
\]

where \( \gamma, \eta \) are Lagrange multipliers.

For the turbulent advection of a passive scalar by a three-dimensional velocity field, the functional \( L \) depends on the passive scalar \( \theta \). For this case the general expressions for the Euler-Lagrange equations are,

\[
\frac{\partial L}{\partial \theta} - \partial_x \frac{\partial L}{\partial (\partial_x \theta)} - \partial_y \frac{\partial L}{\partial (\partial_y \theta)} - \partial_z \frac{\partial L}{\partial (\partial_z \theta)} = 0, \quad (A.8)
\]

Deriving the Euler-Lagrange equations for this case, we find,

\[
\partial_i \left[ \frac{\mathbf{u} \cdot \nabla \theta}{(\mathbf{u} \cdot \nabla \theta)^2} \right] \left[ (\nabla \theta)^2 u_i - (\mathbf{u} \cdot \nabla \theta) \partial_i \theta \right] = -\gamma \theta + \eta \Delta \theta, \quad (A.9)
\]

and these relations should be verified approximately if the minimization principle (A.6) describes the dynamics.

A.2.2 Verification of the conjectures in a stationary turbulent flow

To check the above ideas, the 3D incompressible Navier-Stokes equations are simulated,

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \Delta \mathbf{u} + \mathbf{f} \quad (A.10)
\]

where in the computation the forcing term is given as a negative viscosity forcing term \( \mathbf{f} = -\nu^f \Delta \mathbf{u} \) applied to wavenumbers in the spherical shell between \( k = 1 \) and \( k = 2.5 \). A value of \( \nu^f \) was chosen such that a statistically stationary turbulence was obtained. The flow is computed using a classical spectral method in a periodic box of resolution \( 256 \times 256 \times 256 \) grid points. The initial conditions are given with the method of Rogallo et al. [104] as a Gaussian velocity field. The stationary field has a Taylor-Reynolds number \( R_\lambda \approx 180 \) and integral Reynolds number \( Re = 4899 \). Also, the Prandtl number, defined as the ratio between the kinematic viscosity \( \nu \) and the scalar diffusivity coefficient, is unity: \( Pr = 1 \). A passive
scalar is advected by this flow, governed by

$$\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta = (v/Pr) \Delta \theta. \quad (A.11)$$

The computation becomes quasi-stationary at $t = 2.18$, and at this time we relaunch the computation and inject a freely decaying passive scalar into the velocity field. The initial scalar field is, as the initial velocity field, constituted by Gaussian noise with a spectral distribution,

$$E_\theta(k) = \frac{16}{4.75} \sqrt{\frac{2}{\pi}} k^4 \exp(-2k^2). \quad (A.12)$$

In Figure A.1 we show the temporal evolution of a certain number of volume-averaged quantities. It is observed that energy $\langle u^2 \rangle$ and dissipation $\langle (d_i u_j + d_j u_i)^2 \rangle$ are well conserved due to the forcing of the velocity field, while the variance of $\langle \cos^2 \phi \rangle$ decays to a stable minimum value. Moreover, the variance of scalar-velocity coupling term $\langle (u \cdot \nabla \theta) \rangle$ initially fluctuates, before decaying to zero afterwards.

We will now assess the possible minimum states which are associated with equation (A.9). We therefore consider the different possible global constraints, related to non-zero values of the Lagrange multipliers $\gamma, \eta$, one at a time.

**Scalar variance constraint ($\gamma \neq 0, \eta = 0$)** Other than the dissipation and kinetic energy, we also noted that there seems a small portion of the simulation where the scalar variance $\langle \theta^2 \rangle$ is approximately constant (during $0 \leq t \leq 0.5$), thus we checked the minimization under a constant scalar variance constraint. From the Figure. A.2 we cannot observe an obvious correlation coefficient during this time. Hence unfortunately we have to rule out this hypothesis.

**Scalar dissipation constraint ($\gamma = 0, \eta \neq 0$)** Even though the scalar dissipation is not constant during $0 \leq t \leq 2$, we still checked the minimization under this quantity constraint to see what would happen. Once again Figure. A.3 shows that the correlation coefficients are still inconsistent with the derived equations. Therefore we have to rule out this hypothesis too.

**Advection variance constraint** Despite of the constraints mentioned above, we also considered other quantity such as the advection variance $\langle (u \cdot \nabla \theta)^2 \rangle$, which is approximately constant (during $0 \leq t \leq 1$). Therefore we have a new minimization problem,

$$\min_{\theta}\{\langle \cos^2 \phi \rangle|\langle (u \cdot \nabla \theta)^2 \rangle\}. \quad (A.13)$$
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Fig. A.1 Time history of volume-averaged quantities normalized by their initial values, in a stationary HIT, with a freely decaying scalar.

Fig. A.2 Verifications of minimization principle equation (A.6) in forced turbulence, using global constraint based on the scalar variance. It indicates the correlation coefficient between the left and the right term of equation (A.9) with $\eta = 0$. 
Fig. A.3 Verifications of minimization principle equation (A.6) in forced turbulence, using global constraint based on the scalar dissipation. It indicates the correlation coefficient between the left and the right term of equation (A.9) with $\gamma = 0$.

Fig. A.4 Verifications of minimization principle equation (A.13) in forced turbulence, using global constraint based on the advection variance. This figure indicates the correlation coefficients between the left and the right terms of equation (A.15).
And the functional $L$ in this case is

$$L = \int d\mathbf{x} \{ \cos^2 \phi - \lambda \langle (\mathbf{u} \cdot \nabla \theta)^2 \rangle \}, \quad (A.14)$$

where $\lambda$ are Lagrange multipliers. Deriving the Euler-Lagrange equations for this case, we find,

$$\partial_i \left[ \frac{\mathbf{u} \cdot \mathbf{\nabla} \theta}{(\mathbf{u} \cdot \mathbf{\nabla} \theta)^2} \left[ (\nabla \theta)^2 u_i - (\mathbf{u} \cdot \nabla \theta) \partial_i \theta \right] \right] = \lambda \partial_i [(\mathbf{u} \cdot \nabla \theta) u_i]. \quad (A.15)$$

In the Figure. A.4 we have illustrated the correlation coefficients for the equation (A.15). Unfortunately, we still cannot observe the obvious coefficient during this time. Thus this hypothesis is ruled out.

**Intermediate Conclusion** Until now we have checked a certain number of possible constraints to assess whether the cosine between the velocity and the scalar gradient relaxes to a minimum value under constraints. We considered a a stationary forced HIT. We show that none of the minimization principles seems to be verified. It is obviously not straightforward to show that no such principle exists, but the present results rule out at least the considered possibilities.

### A.2.3 Verification in a freely decaying turbulence

To be sure that the observed results in the previous section are not influenced by the presence of a large-scale forcing, which could possibly disturb the relaxation of the fields, we have also tried to verify the same hypotheses in a decaying isotropic turbulence. The turbulence field data is calculated with the same spectral method, but this time without the forcing term. The initial integral Reynolds number is $\text{Re} = 608$ and the Prandtl number is again $\text{Pr} = 1$. The computation resolution is again $256 \times 256 \times 256$ within a periodic box. The initial condition is generated with the method of Rogallo.

Similar to the previous simulation, we first evaluate the time evolutions of the volume averages in Figure. A.5. We note that at the beginning of the development, the variance of the intersection angle $\langle \cos^2 \phi \rangle$ decays to a constant value within the interval $0 \leq t \leq 2$. The two best conserved quantities appear to be the kinetic energy and the scalar energy $\langle \theta^2 \rangle$. Unlike the forced simulation, we see that most quantities evolve rapidly in time, and constraints of fixed quantities seem not very relevant. However, we have still checked the different cases considered in the previous section, to rule out the possibility that the correlations compatible with a minimization principle are valid in the absence of a forcing term in the Navier-Stokes equations.
Fig. A.5 Time history of the quantities normalized with their initial values, in a decaying turbulence.
A.3 Conclusion

In this chapter, we have attempted to find new minima/maxima principles in 3D turbulence, both decaying and forced. Instead of invariants of the evolution equations, we have considered the global alignment properties of the flow, and we have conjectured that the flow tends to a maximum alignment under given constraints, hereby partially reducing the mean-square advection. Nevertheless, no prove of the validation for the hypothesis we proposed were obtained so far. This means that the alignment between $u$ and $\nabla \theta$ is not maximized by turbulence under the constraints that we have tested, at least at the Reynolds number considered. A maximisation under other constraints might be conceivable.

Fig. A.6 Verifications of minimization principle equation (A.6) in decaying turbulence, using global constraint based on the scalar variance. It indicates the correlation coefficient between the left and the right term of equation (A.9) with $\eta = 0$.

**Minimization with conserved scalar energy** This time the minimization principle to verify is the equation (A.6), with the equivalent equation (A.9) and $\eta = 0$. The results are shown in the Figures. A.6. We note that the coefficient of correlation is also inconsistent with the prediction of the equations. Hence in the decaying turbulence this minima will be ruled out.
Appendix B

Orthonormality of the eigenfunction

The orthonormality of the eigenfunction of the curl have been proven by a series work of Yoshida et al. [28–30]. Yet with the modification of the specific form of the expansion functions $A_{nq}$, we have re-demonstrated the orthonormality in our particular calculation case. Since orthonormality property of the Bessel function is defined as follows

\[ \int_0^1 x J_v(t_n x) J_v(t_n x) \, dx = 0, \ m \neq n \]  

(B.1)

where $t_n$ s ($n = 1, 2, 3, \ldots$) are the zeros (positive roots) of the function. It can also be shown that

\[ \int_0^1 x [J_v(t_n x)]^2 \, dx = \frac{1}{2} [J_{v+1}(t_n)]^2 \]  

(B.2)

More importantly, if $t_n$ s ($n = 1, 2, 3, \ldots$) are NOT the zeros (positive roots) of the function, we have

\[ \int_0^1 x [J_v(t_n x)]^2 \, dx = \frac{1}{2} \left\{ [J_v'(t_n)]^2 + (1 - \frac{v^2}{t_n^2}) [J_v(t_n)]^2 \right\} \]  

(B.3)
With the relations above, if we take the expansion functions into the inner products, we will find that

\[
\frac{1}{V} \int A_{n'q'} \cdot A_{n''q''}^* d^3x = \frac{1}{V} \int I_{n'}^{1/2} I_{n''}^{1/2} \cdot \int_0^{2\pi} d\theta \cdot \int_0^L e^{i(k_{n'} - k_{n''})z} dz.
\]

\[
\left[ k_{n'} k_{n''} \gamma_{n'q'} \gamma_{n''q''} \int_0^a r J_1(\gamma_{n'q'} r) J_1(\gamma_{n''q''} r) dr + \lambda_{n'q'} \lambda_{n''q''} \gamma_{n'q'} \gamma_{n''q''} \int_0^a r J_1(\gamma_{n'q'} r) J_1(\gamma_{n''q''} r) dr + \gamma_{n'q'}^2 \gamma_{n''q''}^2 \int_0^a r J_0(\gamma_{n'q'} r) J_0(\gamma_{n''q''} r) dr \right],
\]

(B.4)

where the region of volume integration is the interior of the circular cylinder \(0 \leq r \leq a, 0 \leq z \leq L_z\). With the equ.3.29 and equ.B.1, and also

\[
\int_0^a r J_0(\gamma_{n'q'} r) J_0(\gamma_{n''q''} r) dr = \frac{\gamma_{n'q'} a J_0(\gamma_{n''q''} a) J_0'(\gamma_{n'q'} a) - \gamma_{n''q''} a J_0(\gamma_{n'q'} a) J_0'(\gamma_{n''q''} a)}{(\gamma_{n''q''})^2 - (\gamma_{n'q'} a)^2},
\]

(B.5)

\[
\int_0^a r J_1(\gamma_{n'q'} r) J_1(\gamma_{n''q''} r) dr = \frac{-\gamma_{n'q'} a J_0(\gamma_{n''q''} a) J_1(\gamma_{n'q'} a) + \gamma_{n''q''} a J_0(\gamma_{n'q'} a) J_1(\gamma_{n''q''} a)}{(\gamma_{n''q''})^2 - (\gamma_{n'q'} a)^2},
\]

(B.6)

we have (if \(n' \neq n''\) or \(q' \neq q''\))

\[
\frac{1}{V} \int A_{n'q'} \cdot A_{n''q''}^* d^3x = 0.
\]

(B.7)

Furthermore, with \(\int A_{nq} \cdot A_{nq}^* d^3x = 1\), equ.3.29, equ.B.2 and equ.B.3, there is

\[
\int_0^a r [J_1(\gamma_{nq} r)]^2 dr = a^2 \int_0^1 x [J_1(\gamma_{nq} ax)]^2 dx = a^2 \frac{1}{2} [J_2(\gamma_{nq} a)]^2,
\]

(B.8)
\[ a \int \left[ J_0(\gamma_{nq}r) \right]^2 dr = a^2 \int_0^a x \left[ J_0(\gamma_{nq}ax) \right]^2 dx = \frac{a^2}{2} \left\{ [J'_0(\gamma_{nq}a)]^2 + (1 - \frac{0^2}{\gamma_{nq}^2 a^2}) [J_0(\gamma_{nq}a)]^2 \right\}, \]  

(B.9)

due to the normalization integral \( I_{nq} \) is

\[ I_{nq} = \frac{1}{V} \pi L_z a^2 \gamma_{nq}^2 \left[ (k_n^2 + \lambda_{nq}^2) J_2^2(\gamma_{nq} a) + \gamma_{nq}^2 J_0^2(\gamma_{nq} a) \right]. \]  

(B.10)

As \( J_1(\gamma_{nq} a) = 0 \) by the no penetration condition, the relation \( J_2(\gamma_{nq} a) = \frac{2}{\gamma_{nq} a} J_1(\gamma_{nq} a) - J_0(\gamma_{nq} a) \) and \( \lambda_{nq}^2 - k_n^2 = \gamma_{nq}^2 \), we could have

\[ I_{nq} = \frac{1}{V} 2\pi L_z a^2 \gamma_{nq}^2 \lambda_{nq}^2 J_0^2(\gamma_{nq} a) = 2 \gamma_{nq} \lambda_{nq}^2 J_0^2(\gamma_{nq} a). \]  

(B.11)

In the article Hudong Chen et al. (1990), they gave the normalization integral \( I_{nmq} \) as

\[ I_{nmq} = 2\pi L_z a^2 \gamma_{nmq}^2 \lambda_{nmq}^2 J_0^2(\gamma_{nmq} a) \times \left( \frac{m \lambda_{nmq}^2 \gamma_{nmq}^2}{k_n} + \lambda_{nmq}^2 (\gamma_{nmq} a)^2 \left( 1 + \frac{m^2}{k_n^2 a^2} \right) \right). \]  

(B.12)

There is no \( \frac{1}{V} \) simply because the inner products in the article didn’t normalize by the volume, the rest part are the same. Hence, if we only substitute \( m = 0 \) here, we have

\[ I_{n0q} = 2\pi L_z a^2 \gamma_{n0q}^2 \lambda_{n0q}^2 J_0^2(\gamma_{n0q} a), \]  

(B.13)

the same as the equ. (B.11), which verified the correctness of our derivations above.

One more thing should be noted here is that we have neglected the \( \frac{1}{V} \) for the inner products, in order to compare with the former researchers’ article [35]. For the sake of consisting with the following calculation, we take account of the normalization of volume here so that

\[ I_{nq} = 2 \gamma_{nq} \lambda_{nq}^2 J_0^2(\gamma_{nq} a). \]  

(B.14)
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[58] A. N. Kolmogorov, “The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers,” JSTOR.


